Feedback Control for Multidimensional Linear Systems and Interpolation Problems for Multivariable Holomorphic Functions

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Abstract: This article provides the connection between feedback stabilization and interpolation conditions for n-D linear systems ($n > 1$). In addition to internal stability, if one demands performance as a design goal, then there results an n-D matrix Nevanlinna-Pick interpolation problem. Application of recent work on Nevanlinna-Pick interpolation on the polydisk yields a solution of the problem for the 2-D case. The same analysis applies in the n-D case ($n > 2$), but leads to solutions which are contractive in a norm (the "Schur-Agler norm") somewhat stronger than the H^∞ norm. This is an analogous version of the connection between the standard H^∞ control problem and an interpolation problem of Nevanlinna-Pick type in the classical 1-D linear time-invariant systems.

Keywords: H^∞ control, model matching problems, multivariable Nevanlinna-Pick interpolation problems, Schur-Agler class, infinite LMI, multidimensional linear systems.

1. Introduction

It is well know that for the case of classical 1-D linear time-invariant systems, the H^∞ control problem can be solved via either state-space analysis in the time domain, or interpolation theory in the transform (frequency) domain. In the latter approach, one goes through coprime factorization to get the Q-parameter; with Q as the new design parameter rather than the controller K, one arrives at a model matching problem. Let F be the performance function, which is affine in Q. Then, with F as the design parameter rather than Q, one has an interpolation problem for F. If one can solve for F in an interpolation problem, then Q and finally K, a desired controller, can be solved. A criterion for internal stability can be expressed directly in terms of F: *K is internally stabilizing for the closed loop system whenever F is stable and satisfies the appropriate interpolation conditions.* Incorporation of a tolerance level on the performance function then leads to an interpolation problem of Nevanlinna-Pick type. (see [15] for more details)

We here extend these results to the case of multidimensional or n-D linear systems where the time-axis is an *integer lattice* rather than "time" in the classical system. Z. Lin studied the (output) feedback stabilization problem for n-D systems (see [17], [18], [19], [20]), and obtained an analogue of the Youla parametrization Q of the set of all stabilizing controllers. In his work, however, Lin did not take the next step of seeking to find a stabilizing controller which optimizes some performance function (i.e., the H^∞-control problem). To the best of our knowledge, the H^∞ control problems in the frequency domain setting have been attacked for the first time in [8] for the output feedback n-D linear system ($n \geq 2$).

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2. Notation and Preliminaries

In this article, we let $\mathbb{R}$ denote the field of real numbers; $\mathbb{R}[\mathbf{z}] = \mathbb{R}[z_1, \ldots, z_d]$ the polynomial ring over $\mathbb{R}$ in d indeterminants $(z_1, \ldots, z_d)$, all of which are complex variables, and $\mathbb{R}(\mathbf{z}) = \mathbb{R}(z_1, \ldots, z_d)$, the field of rational functions which is equal to the quotient field of $\mathbb{R}[\mathbf{z}]$. We denote by $\mathbb{R}_s(\mathbf{z})$ the subset of $\mathbb{R}(\mathbf{z})$ consisting of elements of $r(\mathbf{z}) \in \mathbb{R}(\mathbf{z})$ which are analytic and uniformly bounded on the open unit polydisk $\mathbb{D}^d = \{\mathbf{z}= (z_1, \ldots, z_d)\colon |z_j| < 1\}$ (ℓ^2-gain stable), i.e. $\mathbb{R}_s(\mathbf{z})= \{r(\mathbf{z}) \in \mathbb{R}(\mathbf{z})\colon \sup_{\mathbf{z}\in\mathbb{D}^d} |r(\mathbf{z})| < \infty\}$. We let $\mathbb{R}_{ss}(\mathbf{z})$ denote the set of all real rational functions $r(\mathbf{z})$ having a representation $r(\mathbf{z})= \frac{n(\mathbf{z})}{d(\mathbf{z})}$ with $n(\mathbf{z})$ and $d(\mathbf{z})$ factor-coprime polynomials such that $d(\mathbf{z})$ has no zeros in the closed unit polydisk $\mathrm{cl}(\mathbb{D}^d) = \{\mathbf{z} = (z_1, \ldots, z_d)\colon |z_j| \leq 1\}$ —see Definition 2 of [17]. In the 1-D case ($d = 1$), we have $\mathbb{R}_{ss}(z) = \mathbb{R}_s(z)$ but in general we have only the inclusion $\mathbb{R}_{ss}(\mathbf{z}) \subset \mathbb{R}_s(\mathbf{z})$ due to the possible presence of *nonessential singularities of the second kind* on the boundary $\partial\mathbb{D}^d$ of the polydisk $\mathbb{D}^d$ (see [24] and the references therein). In addition, a number of authors work with the notion of *bounded-input-bounded-output (BIBO) stable* (whereby $r(\mathbf{z}) = \sum_{n\in\mathbb{Z}^d_+} a_n\mathbf{z}^n$ with $\sum_{n\in\mathbb{Z}^d_+} |a_n| < \infty$), but we shall not work with this latter notion here.

We let $\mathbb{R}^{m\times l}(\mathbf{z})$ denote the set of $m \times l$ matrices with entries in $\mathbb{R}(\mathbf{z})$ (i.e., entries are rational functions); $\mathbb{R}^{m\times l}_s(\mathbf{z})$ the set of $m \times l$ matrices with entries in $\mathbb{R}_s(\mathbf{z})$ (i.e., entries are ℓ^2-gain stable real rational functions), and $\mathbb{R}^{m\times l}_{ss}(\mathbf{z})$ the set of $m \times l$ matrices with entries in $\mathbb{R}_{ss}(\mathbf{z})$ (i.e., entries are structurally stable real rational functions). The d-D polynomial is said to be stable if it has no zeros in $\mathrm{cl}(\mathbb{D}^d)$. As we use d rather than n for the number of variables, we shall refer to multivariable systems as d-D rather than n-D systems. Thus a polynomial fraction $r(\mathbf{z}) = \frac{n(\mathbf{z})}{d(\mathbf{z})}$ with stable denominator $d(\mathbf{z})$ gives rise to a structurally stable rational function

$r(\mathbf{z})$.
Throughout this and the succeeding sections, we shall take "stable" to mean "ℓ^2-gain stable". Thus a given rational matrix function X being stable now means that all its matrix entries in $\mathbb{R}_s(\mathbf{z})$ (holomorphic and uniformly bounded on $\mathbb{D}^d$).
For f a (scalar-, vector- or matrix-valued) function which is holomorphic at a point $\mathbf{z}^0 = (z_1^0, \ldots, z_d^0) \in \mathbb{C}^d$, the d dimensional Complex Euclidean space, and $\mathbf{j} = (j_1, \ldots, j_d)$ a d-tuple of nonnegative integers, we denote simply by $\frac{\partial^{|\mathbf{j}|} f}{\partial \mathbf{z}^{\mathbf{j}}}(\mathbf{z})$ the higher order partial derivative

$$\frac{\partial^{|\mathbf{j}|} f}{\partial \mathbf{z}^{\mathbf{j}}}(\mathbf{z}) = \frac{\partial^{j_1+\cdots+j_d} f}{\partial z_1^{j_1} \cdots \partial z_d^{j_d}}(z_1, \ldots, z_d)$$

of f at a point $\mathbf{z} = (z_1, \ldots, z_d)$ of holomorphicity in $\mathbb{C}^d$.
Finally, for f a holomorphic function on $\mathbb{D}^d$, we denote by $\mathcal{Z}(f)$ the zero variety (or zero set) of f

$$\mathcal{Z}(f) = \{\mathbf{z} \in \mathbb{D}^d \colon f(\mathbf{z}) = 0\}.$$

Any subset V' of a variety V of the form $V' = V \setminus A$ where A is a subvariety of V of lower dimension is said to be a *generic* subset of V (see [23]).

3. Equivalence of H^∞ Control, Model Matching and Interpolation Problem

This article applies the results of Lin (see [18], [19], [20]) to provide the connection between the H^∞ control problem in the so-called 1 block case and the multivariable Nevanlinna-Pick interpolation problem under assumption that the plant P admits a doubly coprime factorization (DCF). Under this assumption, one can reformulate the H^∞ control problem into the model matching problem via the Youla parametrization Q.
Suppose now that we arrive at the model matching problem stated as follows: *Given stable rational matrix functions T_1, T_2, and T_3 of compatible sizes, find the stable Q so as to achieve*

$$\min_Q \|T_1 - T_2 Q T_3\| \tag{1}$$

where the norm is the supremum norm over $\mathbb{D}^d$.
Here T_1, T_2 and T_3 are all stable rational matrix function in $\mathbf{z}= (z_1, \ldots, z_d)$ of the appropriate sizes, say, $T_1 \in \mathbb{R}_s^{l\times m}(\mathbf{z})$, $T_2 \in \mathbb{R}_s^{l\times l}(\mathbf{z})$, and $T_3 \in \mathbb{R}_s^{m\times m}(\mathbf{z})$. We shall focus on the so-called *regular 1-block case* (see [15]), i.e., we shall assume that T_2 and T_3 are invertible in $\mathbb{R}^{l\times l}(\mathbf{z})$ and $\mathbb{R}^{m\times m}(\mathbf{z})$ respectively with inverses T_2^{-1} and T_3^{-1} (not necessarily stable) existing and uniformly bounded on the distinguished boundary $\mathbb{T}^d = \{\mathbf{z} = (z_1, \ldots, z_d) \colon |z_j| = 1 \text{ for } j = 1, \ldots, d\}$ of the polydisk.
The performance function F is given by

$$F = T_1 - T_2 Q T_3, \text{ where } Q \in \mathbb{R}^{l\times m}(\mathbf{z}) \tag{2}$$

Since T_1, T_2, T_3 are all stable, if $Q \in \mathbb{R}_s^{l\times m}(\mathbf{z})$, then F is also stable. Conversely, if $F \in \mathbb{R}_s^{l\times m}(\mathbf{z})$, then one can backsolve for Q:

$$Q = T_2^{-1}(T_1 - F)T_3^{-1} \tag{3}$$

Since all quantities on the right hand side are bounded on the distinguished boundary $\mathbb{T}^d$, it follows that Q is bounded on $\mathbb{T}^d$; by the maximum modulus theorem, it then follows that Q is stable once it is guaranteed that Q is holomorphic on $\mathbb{D}^d$. Since T_2^{-1} and T_3^{-1} may or may not be stable, holomorphicity of F on $\mathbb{D}^d$ does not guarantee holomorphicity of Q on $\mathbb{D}^d$ in general, unless some additional interpolation conditions are imposed on F (see Theorem 2). Thus we see that stability for the closed loop system is equivalent to stability of the performance function F together with holomorphicity of the rational matrix function Q given by (3). In case the Model Matching Problem arises from the sensitivity minimization problem for an output feedback configuration, then $l = m$ and we must also impose the well-posedness condition that $\det F$ not vanish identically.
In this section, for convenience, we shall drop the requirement that Q and F be real and rational; these constraints can always be reincorporated at a later stage. With these relaxations, from the discussion above we see that the stability question, formulated with the performance function F taken as the free parameter, reduces to: *characterize those $l \times m$-matrix valued functions F (subject also to the well-posedness constraint* $\det F$ *not identically equal to* 0*) for which (1) F is holomorphic and uniformly bounded on the polydisk $\mathbb{D}^d$, and (2) the function Q given by* (3) *is holomorphic on $\mathbb{D}^d$.*

Theorem 1 (see [8],[9]). *Suppose that we are given an irreducible polynomial $g(\mathbf{z})$ in $\mathbf{z} = (z_1, \ldots, z_d)$ and that k is a given positive integer. Then a necessary and sufficient condition for a scalar-valued holomorphic function f on the polydisk $\mathbb{D}^d$ to have the form*

$$f(\mathbf{z}) = g(\mathbf{z})^k \varphi(\mathbf{z}); \quad \mathbf{z} \in \mathbb{D}^d \tag{4}$$

for some scalar-valued function φ holomorphic on $\mathbb{D}^d$ is that f satisfies the interpolation conditions

$$\left.\frac{\partial^{|\mathbf{j}|} f}{\partial \mathbf{z}^{\mathbf{j}}}\right|_{\mathcal{Z}(g)} = 0 \quad \textit{for } |\mathbf{j}| = 0, 1, \ldots, k-1 \tag{5}$$

on a generic subset of $\mathcal{Z}(g)$.

Remark. In case $g(\mathbf{z}) = z_1$, the interpolation conditions (5) can be collapsed to

$$\left.\frac{\partial^j f}{\partial z_1^j}\right|_{\mathcal{Z}(g)} = 0 \text{ for } j = 0, 1, \ldots, k-1. \tag{6}$$

Indeed, the vanishing of partial derivatives involving the other variables $z_2, \ldots, z_d$ along $\mathcal{Z}(g) = \{\mathbf{z} \in \mathbb{D}^d \colon z_1 = 0\}$ is automatic from the vanishing of f along

$\mathcal{Z}(g)$. More generally, one could do a change of coordinates $\mathbf{z} = (z_1, \ldots, z_d) \mapsto \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d)$ in such a way that $\lambda_1(z) = g(z)$. Then, with respect to these new local coordinates, the interpolation conditions (5) can be reduced to

$$\left.\frac{\partial^j f}{\partial \lambda_1^j}\right|_{\boldsymbol{\lambda}:\, \lambda_1 = 0} = 0 \text{ for } j = 0, 1, \ldots, k-1.$$

This is how the criterion for (4) is expressed in [13]. In the context here such a change of variables is not so useful as it would destroy the rationality of the functions in the interpolation data set.

We now explain the type of interpolation problem to which the model matching problem can be converted in the 1-block case. For $u = 1, \ldots, \eta$, assume that we are given distinct irreducible (scalar) polynomials q_u with zero variety $\mathcal{Z}(q_u)$ having nontrivial intersection with $\mathbb{D}^d$, meromorphic matrix functions G_u and $\widetilde{G}_u$ (of compatible sizes for the interpolation conditions to follow to make sense) with polar divisor not including $\mathcal{Z}(q_u)$, and positive integers k_u. For $v = 1, \ldots, \mu$ assume that similarly we are given distinct irreducible polynomials s_v together with meromorphic matrix functions H_v and $\widetilde{H}_v$ (of compatible sizes) with polar divisor not including $\mathcal{Z}(s_v)$, and positive integers ℓ_v. For each pair of indices (u, v) for which $q_u = s_v =: h_{u,v}$, assume that we are given an additional matrix function R_{uv} meromorphic on a neighborhood of each point of $\mathcal{Z}(h_{u,v})$. The whole aggregate

$$\mathcal{D} = \{q_u, G_u, \widetilde{G}_u, k_u; s_v, H_v, \widetilde{H}_v, \ell_v; R_{uv}\} \qquad (7)$$

we call a *1-block interpolation data set*.

We say that an $l \times m$ matrix-valued function F holomorphic on $\mathbb{D}^d$ satisfies the interpolation conditions associated with $\mathcal{D}$ (denoted by $F \in \mathcal{I}(\mathcal{D})$) if

$$\left.\left\{\frac{\partial^{|\mathbf{i}|}}{\partial \mathbf{z}^{\mathbf{i}}} G_u(\mathbf{z}) F(\mathbf{z})\right\}\right|_{\mathcal{Z}(q_u)} = \left.\left\{\frac{\partial^{|\mathbf{i}|}}{\partial \mathbf{z}^{\mathbf{i}}} \widetilde{G}_u(\mathbf{z})\right\}\right|_{\mathcal{Z}(q_u)}$$

generically on $\mathcal{Z}(q_u)$, for $u = 1, \ldots, \eta$

and $|\mathbf{i}| = 0, 1, \ldots, k_u - 1$, (8)

$$\left.\left\{\frac{\partial^{|\mathbf{j}|}}{\partial \mathbf{z}^{\mathbf{j}}} F(\mathbf{z}) H_v(\mathbf{z})\right\}\right|_{\mathcal{Z}(s_v)} = \left.\left\{\frac{\partial^{|\mathbf{j}|}}{\partial \mathbf{z}^{\mathbf{j}}} \widetilde{H}_v(\mathbf{z})\right\}\right|_{\mathcal{Z}(s_v)}$$

generically on $\mathcal{Z}(s_v)$, for $v = 1, \ldots, \mu$

and $|\mathbf{j}| = 0, 1, \ldots, \ell_v - 1$, and (9)

$$\left.\left\{\frac{\partial^{|\mathbf{l}|}}{\partial \mathbf{z}^{\mathbf{l}}} G_u(\mathbf{z}) F(\mathbf{z}) H_v(\mathbf{z})\right\}\right|_{\mathcal{Z}(h_{u,v})} = \left.\left\{\frac{\partial^{|\mathbf{l}|}}{\partial \mathbf{z}^{\mathbf{l}}} R_{u,v}(\mathbf{z})\right\}\right|_{\mathcal{Z}(h_{u,v})}$$

generically on $\mathcal{Z}(h_{u,v})$ for all pairs of indices (u, v)

with $q_u = s_v$ and for $|\mathbf{l}| = 0, 1, \ldots, k_u + \ell_v - 1$ (10)

Given T_1, T_2 and T_3 (of respective sizes $l \times m$, $l \times l$ and $m \times m$, say) as in the 1-block case of the model matching problem, we associate an interpolation data set $\mathcal{D}$ as follows. Write the $l \times l$ rational matrix valued function $T_2^{-1}(\mathbf{z})$ as $T_2^{-1}(\mathbf{z}) = \left[\frac{p_{ij}}{q_{ij}}(\mathbf{z})\right]_{i,j=1,\ldots,l}$, and consider the set of unstable entries of T_2^{-1}, say $\left\{\frac{p_{i_a,j_a}}{q_{i_a,j_a}}(\mathbf{z})\right\}$ for $a = 1, \ldots, \alpha$. Let $q(\mathbf{z})$ be the least common multiple of $\{q_{i_1,j_1}(\mathbf{z}), \ldots, q_{i_\alpha,j_\alpha}(\mathbf{z})\}$. Also write the $m \times m$ rational matrix valued function $T_3^{-1}(\mathbf{z})$ as $T_3^{-1}(\mathbf{z}) = \left[\frac{r_{ij}}{s_{ij}}(\mathbf{z})\right]_{i,j=1,\ldots,m}$, and consider the set of unstable entries of T_3^{-1}, say $\left\{\frac{r_{i_b,j_b}}{s_{i_b,j_b}}(\mathbf{z})\right\}$ for $b = 1, \ldots, \beta$. Let $s(\mathbf{z})$ be the least common multiple of $\left\{s_{i_1,j_1}(\mathbf{z}), \ldots, s_{i_\beta,j_\beta}(\mathbf{z})\right\}$. Suppose now that $q(\mathbf{z})$ and $s(\mathbf{z})$ can be factored into irreducible polynomials, say $q(\mathbf{z}) = q_1^{k_1}(\mathbf{z}) \cdots q_\eta^{k_\eta}(\mathbf{z})$, where $k_i > 0$ for $i = 1, \ldots, \eta$, and $s(\mathbf{z}) = s_1^{\ell_1}(\mathbf{z}) \cdots s_\mu^{\ell_\mu}(\mathbf{z})$, where $\ell_i > 0$ for $i = 1, \ldots, \mu$, respectively. Then for each $u \in \{1, \ldots, \eta\}$, $T_2^{-1}(\mathbf{z}) = \frac{G_u}{q_u^{k_u}}(\mathbf{z})$, where $G_u(\mathbf{z})$ is a meromorphic matrix function in $\mathbb{D}^d$ with polar divisor not including $\mathcal{Z}(q_u)$, and $q_u^{k_u}$ is an unstable irreducible polynomial with multiplicity k_u. In addition we set $\widetilde{G}_u(\mathbf{z}) = G_u(\mathbf{z}) T_1(\mathbf{z})$, so $\widetilde{G}_u(\mathbf{z})$ is also meromorphic with polar divisor not including $\mathcal{Z}(q_u)$.

Analogously, for each $v \in \{1, \ldots, \mu\}$, $T_3^{-1}(\mathbf{z}) = \frac{H_v}{s_v^{\ell_v}}(\mathbf{z})$, where $H_v(\mathbf{z})$ is a meromorphic matrix function on $\mathbb{D}^d$ with polar divisor not including $\mathcal{Z}(s_v)$. Set $\widetilde{H}_v(\mathbf{z}) = T_1(\mathbf{z}) H_v(\mathbf{z})$, so $\widetilde{H}_v(\mathbf{z})$ is meromorphic with polar divisor not including $\mathcal{Z}(s_v)$. In addition, if q and s have some common factors, say $q_u = s_v$ for some pair of indices u and v, set $h_{u,v} = q_u = s_v$ and $R_{u,v}(\mathbf{z}) = G_u(\mathbf{z}) T_1(\mathbf{z}) H_v(\mathbf{z})$, so $R_{u,v}$ is meromorphic with polar divisor not including $\mathcal{Z}(h_{u,v})$. In this way we have formed an interpolation data set $\mathcal{D}$ as in (7). When $\mathcal{D}$ is formed in this way from T_1, T_2, T_3, let us write $\mathcal{D} = \mathcal{D}_{T_1,T_2,T_3}$.

Now we are ready to state the main theorem, which gives the connection between the model matching and interpolation problems.

Theorem 2. *Let T_1, T_2, T_3 be the data set for a 1-block model matching problem, and let $\mathcal{D}_{T_1,T_2,T_3}$ be the associated interpolation data set as delineated in the previous paragraph. Then a given function F holomorphic on $\mathbb{D}^d$ has the model matching form $F = T_1 - T_2 Q T_3$ for a stable Q if and only if F satisfies the interpolation conditions* (8), (9) *and* (10) *associated with the data set $\mathcal{D}_{T_1,T_2,T_3}$ (i.e., $F \in \mathcal{I}(\mathcal{D}_{T_1,T_2,T_3})$).*

Remark. If one loosens the 1-block assumption on (T_1, T_2, T_3), the model matching form for F is equivalent to interpolation conditions for F on subvarieties of other codimensions, including the possibility of interpolation conditions at isolated points, or, at the opposite extreme, interpolation conditions on the whole of $\mathbb{D}^d$. For the 1-D case ($d = 1$), there are only the two possibilities of codimension equal to 1 or to 0, i.e. interpolation at isolated points or interpolation along the whole unit disk–see e.g. [10] and [11] for a thorough treatment.

4. The Nevanlinna-Pick Interpolation Problem on the Polydisk

We consider the following d-D version of the bitangential Nevanlinna-Pick interpolation problem (for the 1-D version, see e.g. [7], [14]): *given an interpolation data set $\mathcal{D}$ as in* (7)*, find an $l \times m$ matrix-valued function F holomorphic on $\mathbb{D}^d$ satisfying the interpolation conditions* (8), (9), (10) *($F \in \mathcal{I}(\mathcal{D})$) for which in addition*

$$\sup_{\mathbf{z} \in \mathbb{D}^d} \|F(\mathbf{z})\| \leq 1. \tag{11}$$

The class of all matrix functions holomorphic on $\mathbb{D}^d$ and satisfying the norm constraint (11) is often called the *Schur class*; we denote the class of $l \times m$ matrix-valued such functions by $\mathcal{S}_d(\mathbb{C}^m, \mathbb{C}^l)$. For $d > 2$, it turns out that the norm constraint (11) is not so convenient to work with. A closely related class of functions is what we shall call the Schur-Agler class (see [1], [2], [12]), denoted by $\mathcal{SA}_d(\mathbb{C}^m, \mathbb{C}^l)$, namely, the class of all $l \times m$ matrix-valued functions F holomorphic on $\mathbb{D}^d$ such that

$$\sup\{\|F(T_1, \dots, T_d)\|\} \leq 1. \tag{12}$$

where $\mathcal{H}$ is a Hilbert space, $T_j \in \mathcal{L}(\mathcal{H})$, $\|T_j\| < 1$, and $T_i T_j = T_j T_i$ for $i, j = 1, \dots, d$. Here, for $T_1, \dots, T_d$ equal to strict contraction operators on a Hilbert space $\mathcal{H}$, one can define $F(T_1, \dots, T_d)$ as an operator from $\oplus_{j=1}^m \mathcal{H}_j$ to $\oplus_{k=1}^l \mathcal{H}_k$ by

$$\begin{aligned} &F(T_1, \dots, T_d) \\ &= \left(\frac{1}{2\pi i}\right)^d \int_{\mathbb{T}^d} \left[(I - \zeta_1 T_1)^{-1} \cdots (I - \zeta_d T_d)^{-1}\right] \otimes F(\zeta)\, d\zeta. \end{aligned} \tag{13}$$

It is known that $\mathcal{SA}_d(\mathbb{C}^m, \mathbb{C}^l) = \mathcal{S}_d(\mathbb{C}^m, \mathbb{C}^l)$ for $d = 1, 2$, but only $\mathcal{SA}_d(\mathbb{C}^m, \mathbb{C}^l) \underset{\neq}{\subset} \mathcal{S}_d(\mathbb{C}^m, \mathbb{C}^l)$ for $d > 2$. The bitangential Nevanlinna-Pick problem for the class $\mathcal{SA}_d(\mathbb{C}^m, \mathbb{C}^l)$ can be stated as follows: *given an interpolation data set $\mathcal{D}$ as in* (7)*, find an $l \times m$ matrix-valued function F holomorphic on $\mathbb{D}^d$ satisfying the interpolation conditions* (8), (9), (10) *which in addition satisfies* (12), i.e., we seek $F \in \mathcal{I}(\mathcal{D}) \cap \mathcal{SA}_d(\mathbb{C}^m, \mathbb{C}^l)$. By the remarks above, we see that the Schur-Agler-modified bitangential Nevanlinna-Pick interpolation problem ($\mathcal{I}(\mathcal{D})$ with (12)) is exactly the same as the d-D bitangential Nevanlinna-Pick interpolation problem given above ($\mathcal{I}(\mathcal{D})$ with (11)) in case $d = 1, 2$, while, for $d > 2$, a necessary and sufficient condition for solving the Schur-Agler variant gives only a sufficient condition for solving the original version. We shall next discuss results concerning the Schur-Agler version of the bitangential Nevanlinna-Pick interpolation problem.

For the statement of the next result we need one more piece of terminology. For Ω any set and P a function defined on $\Omega \times \Omega$ with value $P(\omega', \omega)$ at $(\omega', \omega) \in \Omega \times \Omega$ equal to an operator from the Hilbert space $\mathcal{K}_\omega$ to the Hilbert space $\mathcal{K}_{\omega'}$, we say that P is a *positive kernel* if for any choice of N points, say $\omega_1, \dots, \omega_N \in \Omega$, and of N vectors $x_1, \dots, x_N$ with $x_i \in \mathcal{K}_{\omega_i}$ for $i = 1, \dots, N$ (where N is any finite number)

$$\sum_{i,j=1}^N \langle P(\omega_i, \omega_j) x_j, x_i \rangle_{\mathcal{K}_{\omega_i}} \geq 0$$

Such objects are closely connected with the theory of reproducing kernel Hilbert spaces (see e.g. [14]). It is well known that an equivalent condition for a given $P(\omega', \omega)$ as above to be a positive kernel is that there be an auxiliary Hilbert space $\mathcal{H}$ and an operator-valued function $\omega \to T(\omega)$ on Ω, where the value $T(\omega)$ at $\omega \in \Omega$ is an operator from $\mathcal{H}$ into $\mathcal{K}_\omega$, such that we have the factorization

$$P(\omega', \omega) = T(\omega') T(\omega)^*.$$

More concretely, one can view an operator-valued function $P(\cdot, \cdot)$ as above as an infinite block-matrix, with rows and columns indexed by the (possibly infinite) set Ω. The condition that P be a positive kernel can then be viewed as an infinite analogue of a positive-definite matrix.

In case no q_u is also an s_v (so the third set of interpolation conditions (10) is vacuous) and the multiplicities k_u and ℓ_v are all equal to 1, we have the following solution of the Schur-Agler variant of the bitangential interpolation problem from [5] (see also [12] for the case of interpolation along finitely many points).

Theorem 3. *Suppose we are given an interpolation data set* (7) *such that, for all $u = 1, \dots, \eta$ and $v = 1, \dots, \mu$, $q_u \neq s_v$ (so the interpolation condition* (10) *is vacuous), $k_u = 1$ and $\ell_v = 1$. Then there exists a matrix-valued function holomorphic on $\mathbb{D}^d$ satisfying the interpolation conditions* (8) *and* (9) *together with the norm constraint* (11) *if and only if there exists d positive kernels $P_1, \dots, P_d$, where*

$$P_j(\omega', \omega)\colon \Omega \times \Omega \to \begin{cases} \mathbb{C}^{l \times l}, & \textit{if } \omega' \in \mathcal{Z}(q_{u'}), \omega \in \mathcal{Z}(q_u) \\ & \textit{for some } u', u; \\ \mathbb{C}^{l \times m}, & \textit{if } \omega' \in \mathcal{Z}(q_{u'}), \omega \in \mathcal{Z}(s_v) \\ & \textit{for some } u', v; \\ \mathbb{C}^{m \times m}, & \textit{if } \omega' \in \mathcal{Z}(s_{v'}), \omega \in \mathcal{Z}(s_v) \\ & \textit{for some } v', v; \end{cases}$$

satisfying the equation

$$\begin{aligned} \sum_{k=1}^d \left[M_k(\omega')^* P_k(\omega', \omega) M_k(\omega) - N_k(\omega')^* P_k(\omega', \omega) N_k(\omega)\right] \\ = X(\omega')^* X(\omega) - Y(\omega')^* Y(\omega) \end{aligned} \tag{14}$$

for all $\omega', \omega \in \Omega$, where

$$\Omega := \left(\bigcup_{u=1}^{\eta} \mathcal{Z}(q_u)\right) \bigcup \left(\bigcup_{v=1}^{\mu} \mathcal{Z}(s_v)\right)$$

and where

$$M_k(\omega) = I_l, N_k(\omega) = \overline{\omega_k} I_l,$$
$$X(\omega) = G_u(\omega)^*, Y(\omega) = \widetilde{G}_u(\omega)^* \quad (15)$$

in case $\omega \in \mathcal{Z}(q_u)$ *for some* $u = 1, \ldots, \eta$,

$$M_k(\omega) = \omega_k I_m, N_k(\omega) = I_m,$$
$$X(\omega) = \widetilde{H}_v(\omega), Y(\omega) = H_v(\omega) \quad (16)$$

in case $\omega \in \mathcal{Z}(s_v)$ *for some* $v = 1, \ldots, \mu$.

Remark. Were it the case that the set Ω in Theorem 3 were finite, then the problem of solving (14) for d positive-definite matrices $P_1, \ldots, P_d$ would be a particular instance of a Linear Matrix Inequality (LMI) for which much research and software is now well developed (see [6], [16]). There does not appear to be much experience developed with infinite LMIs such as (14).

Remark. The scalar case of the interpolation problem with solution sought in Schur-Agler class and with the interpolation nodal varieties all taken to have dimension zero, is simply: *given interpolation nodes* $\mathbf{z}^1, \ldots, \mathbf{z}^n \in \mathbb{D}^d$ *and interpolation values* $w_1, \ldots, w_n \in \mathbb{C}$, *find a scalar function* $F \in \mathcal{SA}_d$ *satisfying the interpolation conditions*

$$F(\mathbf{z}^i) = w_i \text{ for } i = 1, \ldots, n. \quad (17)$$

The original result of Agler [1] on this problem is: *A necessary and sufficient condition for the existence of a scalar function* F *in* $\mathcal{SA}_d$ *satisfying the interpolation conditions* (17) *is that there exist* d *positive-semidefinite* $n \times n$ *matrices* $P_1, \ldots, P_d$ *so that*

$$1 - w_i \overline{w}_j = \sum_{k=1}^{d} (1 - \mathbf{z}_k^i \overline{\mathbf{z}_k^j}) P_k^{i,j} \text{ for } i, j = 1, \ldots, n. \quad (18)$$

This result was extended to the matrix-valued setting (with the interpolation nodal varieties still assumed to be zero-dimensional and without consideration of two-sided interpolation conditions) in [4], [12]. A contour integral formulation which incorporated higher-order interpolation conditions but still at isolated points was solved in [3].

Remark. The other approach for solving the model matching problem is an opreator-theoretic formulation which lends itself to a solution via the recent polydisk Commutant Lifting Theorem. Note that the existence of the Q-parameter in this approach reduces the problem to solving a *Linear Operator Inequality (LOI)*. Complete details appear in [9], [21].

5. Solution of the d-D H^∞ Control Problem

We now return to the H^∞ control problem: *given a d-D plant P, design a stabilizing controller K for which the performance function* $F = F(K)$ *achieves* $\|F\| \leq 1$. The solution procedure of the H^∞ control problem is given in the following theorem.

Theorem 4. *Assume that* P *admits a DCF over* $\mathbb{R}_s(\mathbf{z})$, $P = N_r D_r^{-1} = D_l^{-1} N_l$, *satisfying the Bézout identity and that* Q *is the Youla parametrization. Then the* H^∞ *control problem can be converted to the model matching problem* (1) *with the performance function* F *as in* (2), *where* T_2 *and* T_3 *are square and invertible with inverses uniformly bounded on the distinguished boundary* $\mathbb{T}^d$ *of the polydisk* $\mathbb{D}^d$. *Form the interpolation data set* $\mathcal{D}_{T_1,T_2,T_3}$ *from* T_1, T_2, T_3 *as in Theorem 2, and assume that no* q_u *is also a* s_v *as in the hypotheses of Theorem 3. Then a sufficient (and also necessary if* $d \leq 2$*) condition for the* H^∞*-control problem to have a (not necessarily well-posed) solution is that the Schur-Agler bitangential Nevanlinna-Pick interpolation problem* (8)-(10) *with* (12) *with data set* $\mathcal{D}_{T_1,T_2,T_3}$ *have a solution, or equivalently, that the infinite LMI* (14) *have a positive solution* $P(\omega', \omega)$.
In this case there are explicit realization formulas for solutions F *of the Schur-Agler bitangential Nevanlinna-Pick interpolation problem (see [5]) which meets the* H^∞ *performance criterion* $\|F\| \leq 1$.

Remark. The assumption in the Theorem 4 is the existence of a DCF over $\mathbb{R}_s(\mathbf{z})$ for P. While the existence of such a DCF in general appears not to have been proved, Z. Lin conjectured in [18] that such a DCF (even over $\mathbb{R}_{ss}(\mathbf{z})$) always does exist.
Recently, it has been reported in the work of K. Mori [22] that the model matching problem we consider here is equivalent to the standard H^∞ control problem in the multidimensional linear system. He applies the coordinate-free approach to achieve the result without using the coprime factorization of plants.

6. Open Problems and Discussions

While the procedure described therein does solve the H^∞-control problem, there are a number of remaining issues which are directions for future research.

1. The case where the third coupled interpolation condition (10) appears (i.e., the case where $q_u = s_v$ for some pair of indices (u, v)) remains mysterious.
2. At least to our knowledge, there is missing a reliable analysis on how to solve an infinite LMI; some analysis of whether solutions of a sequence of approximating LMIs can be used to approximate a true solution of the infinite LMI would be helpful. Simple numerical experiments on small-size examples done by the author suggest that one cannot expect to find the solution of the full infinite LMI by approximating with solutions of finite sub-LMIs.
3. It would be of interest to remove the assumption that T_2 and T_3 are invertible, and to handle the case of the general configuration of the standard H^∞ problem.

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