An Itô formula for generalized functionals for fractional Brownian sheet with arbitrary Hurst parameter*

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Abstract

We derive an Itô formula for generalized functionals for the fractional Brownian sheet with arbitrary Hurst parameter $H_1, H_2 \in (0,1)$. As an application, we consider a stochastic integral representation for the local time of the fractional Brownian sheet .

Keywords: Fractional Brownian sheet; Itô formula; Fractional white noise; Local time.

1 Introduction and Preliminaries

The purpose of this paper is to extend the fractional white noise theory of Hu *et al* (2000) to the case where the Hurst parameter takes any numbers H_1 and H_2 in $(0,1)$ not only in $(1/2,1)$. Using this developed theory, we derive an Itô formula for fractional Brownian sheet with arbitrary Hurst parameter. Our result holds for Hurst parameter $H_1, H_2 \in (0,1)$, whereas that of Tudor and Viens (2003) is valid for Hurst parameter $H_1, H_2 \in (1/2,1)$. As an application, we give a stochastic integral representation for the local time.

Let $\mathcal{S}(\mathbb{R}^2)$ be the Schwartz space and the dual space $\Omega := \mathcal{S}'(\mathbb{R}^2)$ be the space of tempered distribution. We consider the white noise space $(\Omega, \mathbf{F}, \mathbb{P})$ as the underlying probability space. By Minlos Theorem, there exists an unique probability measure $\mathbb{P}$ such

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that for all $f \in \mathcal{S}(\mathbb{R}^2)$,

$$\int_{\mathcal{S}'(\mathbb{R}^2)} e^{i<\omega,f>} d\mathbb{P}(\omega) = e^{-(1/2)\|f\|^2_{L^2(\mathbb{R}^2)}}.$$

For $s = (s_1, s_2), t = (t_1, t_2)$ and $x = (x_1, x_2) \in \mathbb{R}^2$, we define $\mathbf{1}_{(s,t)}(x) = \prod_{i=1}^2 \mathbf{1}_{(s_i,t_i)}(x_i)$, where $\mathbf{1}_{(s_i,t_i)}(x_i)$ is given by

$$\mathbf{1}_{(s_i,t_i)}(x_i) = \begin{cases} 1 & \text{for } s_i \leq x_i < t_i \\ -1 & \text{for } t_i \leq x_i < s_i \\ 0 & \text{otherwise.} \end{cases}$$

We introduce two-variable fractional integral and derivative. For $\alpha, \beta \in (0,1)$,

$$(I^\alpha_\pm \otimes I^\beta_\pm f)(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \int_0^\infty u^{\alpha-1} v^{\beta-1} f(x \mp u, y \mp v) du dv,$$

$$(D^\alpha_\pm \otimes D^\beta_\pm f)(x,y) = \frac{\alpha\beta}{\Gamma(1-\alpha)\Gamma(1-\beta)} \int_0^\infty \int_0^\infty \frac{(\Delta^2_{\mp(u,v)} f)(x,y)}{u^{\alpha+1} v^{\beta+1}} du dv,$$

$$(I^\alpha_\pm \otimes D^\beta_\pm f)(x,y) = \frac{\beta}{\Gamma(\alpha)\Gamma(1-\beta)} \int_0^\infty \int_0^\infty \frac{u^{\alpha-1}[f(x \mp u, y) - f(x \mp u, y \mp v)]}{v^{\beta+1}} du dv,$$

$$(D^\alpha_\pm \otimes I^\beta_\pm f)(x,y) = \frac{\alpha}{\Gamma(1-\alpha)\Gamma(\beta)} \int_0^\infty \int_0^\infty \frac{v^{\beta-1}[f(x, y \mp v) - f(x \pm u, y \mp v)]}{u^{\alpha+1}} du dv,$$

where $\otimes$ denotes the tensor product of linear operators and

$$(\Delta^2_{\pm(h_1,h_2)} f)(x_1, x_2) = f(x_1, x_2) - f(x_1 \pm h_1, x_2) - f(x_1, x_2 \pm h_2) + f(x_1 \pm h_1, x_2 \pm h_2).$$

For $H_1, H_2 \in (0,1)$, we define

$$(A_\pm \otimes B_\pm)^{H_1 H_2} f = \begin{cases} C(H)(D^{(1/2)-H_1}_\pm \otimes D^{(1/2)-H_2}_\pm) f & \text{for } 0 < H_1 < \frac{1}{2}, 0 < H_2 < \frac{1}{2} \\ C(H)(D^{(1/2)-H_1}_\pm \otimes I^{H_2-(1/2)}_\pm) f & \text{for } 0 < H_1 < \frac{1}{2}, \frac{1}{2} < H_2 < 1 \\ C(H)(I^{H_1-(1/2)}_\pm \otimes D^{(1/2)-H_2}_\pm) f & \text{for } \frac{1}{2} < H_1 < 1, 0 < H_2 < \frac{1}{2} \\ C(H)(I^{H_1-(1/2)}_\pm \otimes I^{H_2-(1/2)}_\pm) f & \text{for } \frac{1}{2} < H_1 < 1, \frac{1}{2} < H_2 < 1 \\ f & \text{for } H_1 = H_2 = \frac{1}{2}. \end{cases}$$

The following Theorem is the two-parameter version of Theorem 2.2 given in Bender (2003).

Theorem 1 *A continuous version of* $< \cdot, (A_- \otimes B_-)^{H_1,H_2} \mathbf{1}_{(0,t)} >$ *is a fractional Brownian sheet with arbitrary Hurst parameter* $H_1, H_2 \in (0,1)$.

From Theorem 1, approximating by step functions we easily see that

$$< \omega, (A_- \otimes B_-)^{H_1,H_2} f > = \int_{\mathbb{R}^2} f(t_1, t_2) dB^{H_1,H_2}_{t_1,t_2}(\omega) \text{ for } (A_- \otimes B_-)^{H_1,H_2} f \in L^2(\mathbb{R}^2).$$

2 Generalized functionals of fractional Brownian sheet

Let $\mathbf{H}_n(x), n = 0, 1, \cdots$, be Hermite polynomials defined by $\mathbf{H}_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$ and $h_n(x), n = 0, 1, \cdots$, be Hermite functions $h_n(x) = \frac{1}{\pi^{1/4}(n!2^n)^{1/2}} \mathbf{H}_n(x) e^{-x^2/2}$. Let $\mathbb{N} = \{1, 2, \cdots\}$. Define the operator $A = -\frac{d^2}{dx^2} + x^2 + 1$. For each $p \in \mathbb{Z}$, we introduce the norm $|f|_p = |(A^{\otimes 2})^p f|_0$, where $|\cdot|_0$ is the $L^2(\mathbb{R}^2)$-norm. The set $\{h_n \otimes h_m, n, m = 1, 2, \cdots\}$ is an orthonormal basis for $L^2(\mathbb{R}^2)$. Hence $|f|_p$ is given by

$$|f|_p^2 = \sum_{n,m=1}^{\infty} (2n+2)^{2p}(2m+2)^{2p}(f, h_n \otimes h_m),$$

where $(\cdot, \cdot)$ is the inner product of $L^2(\mathbb{R}^2)$. Define $\mathcal{S}_p(\mathbb{R}^2)$ as the completion of $\mathcal{S}(\mathbb{R}^2)$ with respect to the norm $|\cdot|_p$ and denote by $\mathcal{S}_{-p}(\mathbb{R}^2)$ its dual.

Let $(L^2) = L^2(\Omega, \mathbb{P})$. The multiple stochastic integrals I_n are interpreted with respect to the 2-parameter Wiener process W. For every $\varphi \in (L^2)$, there exist (uniquely determined) functions $f_n \in \hat{L}^2(\mathbb{R}^{2n})$ such that $\varphi(\omega) = \sum_{n=0}^{\infty} I_n(f_n)$. Here $\hat{L}^2(\mathbb{R}^{2n})$ denotes the space of symmetric functions in $L^2(\mathbb{R}^{2n})$. Moreover we have $\|\varphi\|_0^2 := \mathbb{E}[\varphi] = \sum_{n=0}^{\infty} n!|f_n|_0^2$. We define the second quantization operator $\Gamma(A)\varphi \in (L^2)$ by $\Gamma(A)\varphi = \sum_{n=0}^{\infty} I_n((A^{\otimes 2})^{\otimes n} f_n)$. For each $p \in \mathbb{N}$, define $\|\varphi\|_p = \|\Gamma(A)^p \varphi\|_0$ where $\|\cdot\|_0$ is the (L^2)-norm. Let $(\mathcal{S}_p) = \{\varphi \in (L^2) : \|\varphi\|_0 < \infty\}$ be a Hilbert space with norm $\|\cdot\|_p$. Define $(\mathcal{S})$ by the projective limit of $\{(\mathcal{S}_p) : p \in \mathbb{N}\}$ called the space of *test functions*. The dual space $(\mathcal{S})^*$ of $(\mathcal{S})$ is called a space of *generalized functions* (or *Hida distribution*). Also $(\mathcal{S})^* = \cup_p (\mathcal{S}_p)^*$ and the norm on the dual space $(\mathcal{S}_p)^*$ of $(\mathcal{S}_p)$ is given by $\|\varphi\|_{-p} = \|\gamma(A)^{-p}\varphi\|_0$, $p > 0$. The dual action is denoted by $\ll \cdot, \cdot \gg$. Then $B_{t_1,t_2}^{H_1,H_2}$ is differentiable in $(\mathcal{S})^*$.

Theorem 2 *The fractional Brownian sheet* $B_{t_1,t_2}^{H_1,H_2} =< \cdot, (A_- \otimes B_-)^{H_1,H_2} \mathbf{1}_{(0,t)} >$ *has the second order partial derivatives*

From the results in Section 7.1 in Kuo (1996), the generalized functionals of $B_{t_1,t_2}^{H_1,H_2}$ is given by

$$\begin{aligned} F(B_{t_1,t_2}^{H_1,H_2}) &= \frac{1}{\sqrt{2\pi} t_1^{H_1} t_2^{H_2}} \sum_{n=0}^{\infty} \frac{1}{n! t_1^{2nH_1} t_2^{2nH_2}} \int_{\mathbb{R}} F(y) \phi_{t,H,n}(y) dy \\ &\quad \times I_n(((A_- \otimes B_-)^{H_1,H_2} \mathbf{1}_{(0,t)})^{\otimes n}). \end{aligned} \tag{2.1}$$

By Theorem 7.3 in Kuo (1996), we can obtain the S-transform of generalized functionals of $B_{t_1,t_2}^{H_1,H_2}$.

Theorem 3 *Let $H_1, H_2 \in (0,1)$, $F \in \mathcal{S}'(\mathbb{R})$ and $t_1, t_2 > 0$. Then the S-transform of generalized functionals of $B^{H_1,H_2}_{t_1,t_2}$ is given by*

$$S(F(B^{H_1,H_2}_{t_1,t_2}))(\xi) = \frac{1}{\sqrt{2\pi}t_1^{H_1}t_2^{H_2}} \int_{\mathbb{R}} F(y)$$

$$\times \exp\left[-\frac{1}{2t_1^{2H_1}t_2^{2H_2}} \left(y - \int_0^{t_1}\int_0^{t_2} ((A_+ \otimes B_+)^{H_1,H_2}\xi)(s_1,s_2)ds_1ds_2 \right)^2 \right] dy. \quad (2.2)$$

We define the various integrals appearing Itô formula for fractional Brownian sheet and shall use the following notations:

$$\int_a^b \int_c^d F^{(1)}((t_1,t_2), B^{H_1,H_2}_{t_1,t_2}) dB^{H_1,H_2}_{t_1,t_2}$$

$$:= \int_a^b \int_c^d F^{(1)}((t_1,t_2), B^{H_1,H_2}_{t_1,t_2}) \diamond W^{H_1,H_2}_{t_1,t_2} dt_1 dt_2, \quad (2.3)$$

$$\int_a^b \int_c^d F^{(2)}((t_1,t_2), B^{H_1,H_2}_{t_1,t_2}) d_{t_1} B^{H_1,H_2}_{t_1,t_2} d_{t_2} B^{H_1,H_2}_{t_1,t_2}$$

$$:= \int_a^b \int_c^d F^{(2)}((t_1,t_2), B^{H_1,H_2}_{t_1,t_2}) \diamond \left(\frac{\partial}{\partial t_1} B^{H_1,H_2}_{t_1,t_2} \diamond \frac{\partial}{\partial t_2} B^{H_1,H_2}_{t_1,t_2} \right) dt_1 dt_2, \quad (2.4)$$

$$2H_1 \int_a^b \int_c^d F^{(3)}((t_1,t_2), B^{H_1,H_2}_{t_1,t_2}) t_1^{2H_1-1} t_2^{2H_2} dt_1 d_{t_2} B^{H_1,H_2}_{t_1,t_2}$$

$$:= \int_a^b \int_c^d F^{(3)}((t_1,t_2), B^{H_1,H_2}_{t_1,t_2}) \diamond \left(\frac{\partial}{\partial t_{21}} t_1^{2H_1} \frac{\partial}{\partial t_2} B^{H_1,H_2}_{t_1,t_2} \right) t_2^{2H_2} dt_1 dt_2, \quad (2.5)$$

$$2H_2 \int_a^b \int_c^d F^{(3)}((t_1,t_2), B^{H_1,H_2}_{t_1,t_2}) t_1^{2H_1} t_2^{2H_2-1} d_{t_1} B^{H_1,H_2}_{t_1,t_2} dt_2$$

$$:= \int_a^b \int_c^d F^{(3)}((t_1,t_2), B^{H_1,H_2}_{t_1,t_2}) \diamond \left(\frac{\partial}{\partial t_1} B^{H_1,H_2}_{t_1,t_2} \frac{\partial}{\partial t_2} t_2^{2H_2} \right) t_1^{2H_1} dt_1 dt_2. \quad (2.6)$$

3 Itô formula and local time

In this section we prove the Itô formula for generalized functionals of a fractional Brownian sheet with Hurst parameter $H_1, H_2 \in (0,1)$.

Theorem 4 *Let $F \in C^2([a,b]\times[c,d], \mathcal{S}'(\mathbb{R}))$ such that for $i = 1,2,3,4$,*

$$F^{(i)} = \frac{\partial^i}{\partial x^i} F : [a,b]\times[c,d] \to \mathcal{S}'(\mathbb{R})$$

continuous where $F^{(i)}$ are distribution derivative of F. Then for any $0 < a \le b$ and $0 < c \le d$ we have that in $(\mathcal{S})^*$ the equation holds

$$\begin{aligned}
& F((b,d), B^{H_1,H_2}_{b,d}) - F((a,d), B^{H_1,H_2}_{a,d}) - F((b,c), B^{H_1,H_2}_{b,c}) + F((a,c), B^{H_1,H_2}_{a,c}) \\
= & \int_a^b \int_c^d \frac{\partial^2}{\partial t_1 \partial t_2} F((t_1,t_2), B^{H_1,H_2}_{t_1,t_2}) dt_1 dt_2 + \int_a^b \int_c^d F^{(1)}((t_1,t_2), B^{H_1,H_2}_{t_1,t_2}) dB^{H_1,H_2}_{t_1,t_2} \\
& + H_1 \int_a^b \int_c^d \frac{\partial}{\partial t_2} F^{(2)}((t_1,t_2), B^{H_1,H_2}_{t_1,t_2}) t_1^{2H_1-1} dt_1 dt_2 \\
& + \int_a^b \int_c^d \frac{\partial^2}{\partial t_2} F^{(1)}((t_1,t_2), B^{H_1,H_2}_{t_1,t_2}) d_{t_1} B^{H_1,H_2}_{t_1,t_2} dt_2 \\
& + H_2 \int_a^b \int_c^d \frac{\partial}{\partial t_1} F^{(2)}((t_1,t_2), B^{H_1,H_2}_{t_1,t_2}) t_2^{2H_2-1} dt_1 dt_2 \\
& + \int_a^b \int_c^d \frac{\partial^2}{\partial t_1} F^{(1)}((t_1,t_2), B^{H_1,H_2}_{t_1,t_2}) dt_1 d_{t_2} B^{H_1,H_2}_{t_1,t_2} \\
& + \int_a^b \int_c^d F^{(2)}((t_1,t_2), B^{H_1,H_2}_{t_1,t_2}) d_{t_1} B^{H_1,H_2}_{t_1,t_2} d_{t_2} B^{H_1,H_2}_{t_1,t_2} \\
& + 2H_1H_2 \int_a^b \int_c^d F^{(2)}(B^{H_1,H_2}_{t_1,t_2}) t_1^{2H_1-1} t_2^{2H_2-1} dt_1 dt_2 \\
& + H_1 \int_a^b \int_c^d F^{(3)}((t_1,t_2), B^{H_1,H_2}_{t_1,t_2}) t_1^{2H_1-1} t_2^{2H_2} dt_1 d_{t_2} B^{H_1,H_2}_{t_1,t_2} \\
& + H_2 \int_a^b \int_c^d F^{(3)}((t_1,t_2), B^{H_1,H_2}_{t_1,t_2}) t_1^{2H_1} t_2^{2H_2-1} d_{t_1} B^{H_1,H_2}_{t_1,t_2} dt_2 \\
& + H_1H_2 \int_a^b \int_c^d F^{(4)}((t_1,t_2), B^{H_1,H_2}_{t_1,t_2}) t_1^{4H_1-1} t_2^{4H_2-1} dt_1 dt_2. \qquad (3.7)
\end{aligned}$$

Here we assume that the integrals appearing in (3.7) are integrable.

As an application of the generalized Itô formula, we show the stochastic integral representation of local time of a fractional Brownian sheet with arbitrary Hurst parameter.

Define the local time $L^{H_1,H_2}_{t_1,t_2}(a)$, $a \in \mathbb{R}$, of the fractional Brownian sheet $B^{H_1,H_2}_{t_1,t_2}$ as the density of the occupation measure

$$\Gamma^{H_1,H_2}_{t_1,t_2}(B) = 2H_1H_2 \int_0^{t_1}\int_0^{t_2} 1_B(B^{H_1,H_2}_{s_1,s_2}) s_1^{4H_1-1} s_2^{4H_2-1} ds_1 ds_2, \ \mathcal{B}(\mathbb{R}). \tag{3.8}$$

Then we have the following Theorem.

Theorem 5 *Let $H_1, H_2 \in (0,1)$, $a \in \mathbb{R}$ and $t_1, t_2 > 0$. Then we have*

$$\begin{aligned} L^{H_1,H_2}_{t_1,t_2}(a) &= \frac{1}{6}(B^{H_1,H_2}_{b,d} - a)^2 |B^{H_1,H_2}_{b,d} - a| \\ &\quad -\frac{1}{2}\int_0^b\int_0^d (B^{H_1,H_2}_{t_1,t_2} - a)|B^{H_1,H_2}_{t_1,t_2} - a| dB^{H_1,H_2}_{t_1,t_2} \\ &\quad -\int_0^b\int_0^d |B^{H_1,H_2}_{t_1,t_2} - a| d_{t_1}B^{H_1,H_2}_{t_1,t_2} d_{t_2}B^{H_1,H_2}_{t_1,t_2} \\ &\quad -2H_1H_2\int_0^b\int_0^d |B^{H_1,H_2}_{t_1,t_2} - a| t_1^{2H_1-1} t_2^{2H_2-1} dt_1 dt_2 \\ &\quad -H_1\int_0^b\int_0^d sgn(B^{H_1,H_2}_{t_1,t_2} - a) t_1^{2H_1-1} t_2^{2H_2} dt_1 d_{t_2}B^{H_1,H_2}_{t_1,t_2} \\ &\quad -H_2\int_0^b\int_0^d sgn(B^{H_1,H_2}_{t_1,t_2} - a) t_1^{2H_1} t_2^{2H_2-1} d_{t_1}B^{H_1,H_2}_{t_1,t_2} dt_2. \end{aligned} \tag{3.9}$$

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