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The Change Point Analysis in Time Series Models

Sangyeol Lee
Seoul National Univsersity

Abstract

We consider the problem of testing for parameter changes in time series models based on a cusum test. Although the test procedure is well-established for the mean and variance in time series models, a general parameter case has not been discussed in the literature. Therefore, here we develop a cusum test for parameter change in a more general framework. As an example, we consider the change of the parameters in an RCA(1) model and that of the autocovariances of a linear process. We also consider the variance change test for unstable models with unit roots and GARCH models.

1. Introduction

Since the paper of Page (1955), the problem of testing for a parameter change has been an important issue among statisticians. Originally, the problem began with iid samples; see Hinkley (1971), Brown, Durbin and Evans (1975), Zacks (1983), Csorgo and Horvath (1988), Krishnaiah and Mia (1988), Inclan and Tiao (1994), and it moved naturally into the time series context since economic time series often exhibit prominent evidence for structural change in the underlying model; see, for example, Wichern, Miller and Hsu (1976), Picard (1985), Kramer, Ploberger and Alt (1988), Tang and MacNeil (1993), Kim, Cho and Lee (2000), Lee and Park (2001), and the papers cited therein. If the random observations are iid and follow a specific parametric model, one may consider utilizing a likelihood ratio method as in Csorgo and Horvath (1997). However, the method is no longer applicable if the underlying distribution is completely unknown. In such a case, a nonparametric approach should be considered as an alternative. From this viewpoint, here we pay attention to the cusum method for testing for parameter change.

The cusum method is easy to handle and useful for detecting the locations of change points as seen in Inclan and Tiao (1994). In particular, it has been utilized for testing for a change of mean, variance and distribution function (cf. Bai, 1994). A convenience of the method lies in the fact that the sample mean, variance and distribution function are all expressed as the sum of iid random variables, and the convergence result of the cusum test statistic is easily obtained. In fact, Nyblom (1989) considered a sort of cusum method to handle the change point problem for parameters other than the mean and variance. However, the test procedure assumes that the underlying distribution of observations belongs to a known distribution family, and the test statistic is based on estimators relying on the underlying distribution. Unlike in his approach, here we pursue a cusum test, which is totally free from assumptions about the underlying distribution. In fact, our cusum test can be constructed by imitating the one for a mean change in the iid sample. Conventionally, theestimators of a target parameter after normalization are expressed as the sum of the average of iid r.v.'s and a negligible term. The basic idea is then to view the change problem for the parameter as the one for the expected value of the r.v.'s in that expression since a change of parameter would affect the expected value. Following this idea, one can easily construct the cusum test statistic. The details are presented in Section 2.

2 Cusum test

Here we explain how the cusum test is constructed. As an illustration, we consider the test for a mean change in an iid sample based on the following process

$$\begin{aligned} U_n(s) &= \frac{1}{\sqrt{n}\sigma}\left(\sum_{t=1}^{[ns]} x_t - ([ns]/n)\sum_{t=1}^{n} x_t\right) \\ &= \frac{[ns]}{\sqrt{ns}}(\hat{\mu}_{[ns]} - \hat{\mu}_n),\ 0 \leqq s \leqq 1 \end{aligned} \tag{1}$$

where $x_1, ..., x_n$ are iid with mean μ and variance σ^2, and $\hat{\mu}_n = n^{-1}\sum_{t=1}^{n} x_t$. It is well-known that $\{U_n\}$ converges weakly to a standard Brownian bridge, and a test is performed based on the convergence result. Similar reasoning can be adopted for the more general case.

Suppose that one is interested in testing for a change of θ based on a consistent estimator $\hat{\theta}_n$. As with the MLE (maximum likelihood estimator), usually $\hat{\theta}_n$ can be written as

$$\hat{\theta}_n - \theta = n^{-1}\sum_{t=1}^{n} l_t + o_p\left(\frac{1}{\sqrt{n}}\right)$$

(cf. Durbin, 1973), where $l_t := l_t(\theta)$ are iid random variables with zero mean and a second moment. If the l_t are observable as in (1), one can construct a cusum test based on

$$\begin{aligned} V_n(s) &:= n^{-1/2}(El_t^2)^{-1/2}\left(\sum_{t=1}^{[ns]} l_t - ([ns]/n)\sum_{t=1}^{n} l_t\right) \\ &\cong \frac{[ns]}{\sqrt{n}(El_t^2)^{1/2}}(\hat{\theta}_{[ns]} - \hat{\theta}_n), 0 \leqq \theta \leqq 1 \end{aligned} \tag{2}$$

However, generally the l_t are unobservable, and there must be a justification for having the argument in (2). In time series models, $\{l_t\}$ usually forms a sequence of stationary martingale differences (cf. Sections 3 and 4).

Now, let us consider the stationary time series $\{x_t; t=0,\pm 1,\pm 2,\ldots\}$, and let $\theta=(\theta_1,\ldots,\theta_J)'$ be the parameter vector, which will be examined for constancy, e.g. the mean, variance, autocovariances, etc. Here, we wish to test the following hypotheses based on the estimators $\hat{\theta}_n$.

$H_0 : \theta$ does not change for $x_1,\ldots,x_n$. vs.

H_1 : not H_0

Let $\hat{\theta}_k$ be the estimator of θ based on $x_1,\ldots,x_n$. As we saw in (2), we investigate the differences $\hat{\theta}_k-\hat{\theta}_n$, $k=1,\ldots,n$, for constructing a cusum test. The details are addressed below.

Suppose that $\hat{\theta}_k$ obtained from $x_1,\ldots,x_n$, satisfies the following

$$\sqrt{k}(\hat{\theta}_k-\theta)=\frac{1}{\sqrt{k}}\sum_{t=1}^{k} l_t+\Delta_k, \tag{3}$$

where $l_t := l_t(\theta)=(l_{1,t},\ldots,l_{J,t})'$ forms stationary martingale differences with respect to a filtration $\{F_t\}$, namely, for every t,

$$E(l_t|F_{t-1})=0 \ \ a.s., \tag{4}$$

and $\Delta_k=(\Delta_{1,k},\ldots,\Delta_{J,k})'$.

Let $\Gamma= Var(l_t)$ be the covariance matrix of l_t. Assuming that Γ is nonsingular, we define the normalized martingale differences $\xi_t := \Gamma^{-1/2} l_t$. Note that $\xi_t=(\xi_{1,t},\ldots,\xi_{J,t})'$ has uncorrelated components and satisfies (4). Thus if we put

$$\xi_{n,t}=(\xi_{1,n,t},\ldots,\xi_{J,n,t})' := n^{-1/2}\xi_t \tag{5}$$

it holds that

$$\sum_{t=1}^{[ns]}\xi_{n,t} \xrightarrow{w} W_J(s) \tag{6}$$

in the $D^J[0,1]$ space (cf. Billingsley, 1968), where $W_J(s)=(\mathbb{W}_1(s),\ldots,\mathbb{W}_J(s))'$ denotes a J-dimensional standard Brownian motion, since the following conditions are satisfied (cf. Gaenssler and Haeusler, 1986, page 311)

(1) For $j=1,\ldots,J$ and $s\in[0,1]$, $\displaystyle\sum_{t=1}^{[ns]}E(\xi^2_{j,n,t}|F_{t-1}) \xrightarrow{P} s.$ (7)

(2) For $j=1,\dots,J$ and $\epsilon>0 \sum_{t=1}^{n} E(\xi_{j,n,t}^2 I(|\xi_{j,n,t}|>\epsilon)|F_{t-1}) \xrightarrow{P} 0.$ (8)

Now, suppose that for each $j=1,\dots,J$,

$$\max_{1\leq k\leq n}\frac{\sqrt{k}}{\sqrt{n}}|\Delta_{j,k}| = o_p(1). \tag{9}$$

Then from (3), (6) and (9), we have that

$$\frac{[ns]}{\sqrt{n}}\Gamma^{-1/2}(\hat{\theta}_{[ns]}-\theta) = \sum_{t=1}^{[ns]}\xi_{n,t} + \Gamma^{-1/2}\frac{\sqrt{[ns]}}{\sqrt{n}}\Delta_{[ns]} \xrightarrow{w} W_J(s), \tag{10}$$

and consequently,

$$\frac{[ns]}{\sqrt{n}}\Gamma^{-1/2}(\hat{\theta}_{[ns]}-\hat{\theta}_n) \xrightarrow{w} W_J^o(s), \tag{11}$$

where $W_J^o(s) = (\mathbb{W}_1^o(s),\dots,\mathbb{W}_J^o(s))'$ is a J-dimensional standard Brownian bridge. The following is a direct result of (3)-(11).

Theorem 1 Define the test statistic T_n by

$$T_n = max_{J\leq k\leq n}\frac{k^2}{n}(\hat{\theta}_k-\hat{\theta}_n)'\Gamma^{-1}(\hat{\theta}_k-\hat{\theta}_n).$$

Suppose that conditions (6) and (9) hold. Then, under H_0,

$$T_n \xrightarrow{w} sup_{0\leq s\leq 1}\sum_{j=1}^{J}(W_j^o(s))^2.$$

We reject H_0 if T_n is large.

The cusum test can be applied to the parameter change test for autocorrelations for stationary linear processes, unstable models with several unit roots, and GARCH models. In the lecture, the author will provide the audience with detailed results.

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