On the Properties of OWA Operator Weighting Functions with Constant Value of Orness

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Abstract

In this paper, we present analytic forms of the ordered weighted averaging (OWA) operator weighting functions, each of which has properties of rank-based weights and a constant level of orness, irrespective of the number of objectives considered. These analytic forms provide significant advantages for generating OWA weights over previously reported methods. First, OWA weights can be efficiently generated by use of proposed weighting functions without solving a complicated mathematical program. Moreover, convex combinations of these specific OWA operators can be used to generate OWA operators with any predefined values of orness once specific values of orness are *a priori* stated by decision maker. Those weights have a property of constant level of orness as well. Finally, OWA weights generated at a predefined value of orness make almost no numerical difference with maximum entropy OWA weights in terms of dispersion.

1. Introduction

Yager [10] introduced the ordered weighted averaging (OWA) operator to provide a method for aggregating multiple inputs that lie between the *max* and *min* operators. As the term 'ordered' implies, the OWA operator pursues a nonlinear aggregation of objects considered. In the short time since their first appearance, the OWA operators have been used in an astonishingly wide range of applications in the fields including neural networks [12][13], database systems [9], fuzzy logic controllers [11][15], a group decision making with linguistic assessments [6], data mining [8] and so on. The main reason for this is their great flexibility to model a wide variety of aggregators, as their nature is defined by a weighting vector, and not by a single parameter [1]. By appropriately selecting the weighting vector, we can model different kinds of relationships between the criteria aggregated. Recently, Xu and Da [16] present a survey of the main aggregation operators that encompass a broad range of existing operators (more than 20 aggregators). It is clear that actual results of aggregation performed by OWA operators depend upon the forms of the weighting vectors, which play a key role in aggregation process. Filev and Yager [3] present a way of obtaining the weights associated with the OWA aggregation in the situation when we have observed data on the arguments and the aggregated value (see [2][4][5][7] for the methods of determining OWA weights).

Another appealing point was the introduction of the concept of *orness* and the definition of an orness measure that could establish how 'orlike' a certain operator is. Thus the measure can be interpreted as the mode of decision making by conferring the semantic meaning to the weights used in aggregation process. Moreover, Yager [10] used a measure of entropy to gauge a degree of utilization of information in the sense that each of weighting vectors considered can be different to each other in terms of dispersion although they have the same value in terms of orness.

In this paper, we present four analytic forms of OWA operator weighting functions, each of which has properties of rank-based weights and constant degree of orness, irrespective of the number of objectives considered. These analytic forms provide significant advantages for generating OWA weights over previously reported methods. These findings will be validated by several theorems and corollaries in Section 2 and concluding remarks follow in Section 3.

2. Analytic forms of OWA operator weights with constant level of orness and their properties

An OWA operator [10] of dimension n is a mapping f: $R^n \rightarrow R$ that has an associated weighting n vector $W=(w_1,w_2,\cdots,w_n)^T$ such that $w_i \in [0,1]$ for $i \in I=\{1,2,\cdots,n\}$ and $\sum_{i\in I} w_i=1$. Under this type of operator the function value f determines the aggregated value of arguments $a_1,a_2,\cdots,a_n$ in such a manner that $f(a_1,a_2,\cdots,a_n)=\sum_{i\in I} w_i b_i$, where b_i is the ith largest element in the collection, thus satisfying the relation $Min_i[a_i] \leq f(a_1,a_2,\cdots,a_n) \leq Max_i[a_i]$. In [10], Yager introduced two characterizing measures associated with weighting vector W of an OWA operator. The measure of *orness* of the aggregation is defined as

$$orness(W)=\frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i$$

and it characterizes the degree to which the aggregation is like an *or* operation. Furthermore Yager [10] associates with any OWA vector W a measure of dispersion of entropy. The procedure for obtaining the associated OWA weights then becomes that of solving a mathematical programming problem which selects the weights that have the maximum entropy (dispersion) while satisfying the requirement on the optimism [7]. With regard to the measure of orness, we present, via some theorems and corollaries, analytic forms of OWA weights having constant level of orness.

Theorem 1: The orness of OWA weights in (1) is ¾ irrespective of n when n is referred to as the number of objectives.

$$w_i=(1/n)\sum_{j=i,\cdots,n}(1/j) \quad (1)$$

Proof:

$$orness(W)=\frac{1}{n-1}\sum_{i=1}^{n}\left((n-i)\frac{1}{n}\sum_{j=i}^{n}\frac{1}{j}\right)=\frac{1}{n(n-1)}\left(\sum_{i=1}^{n}\left(n\cdot\sum_{j=i}^{n}\frac{1}{j}\right)-\sum_{i=1}^{n}\left(i\cdot\sum_{j=i}^{n}\frac{1}{j}\right)\right)$$

$$=\frac{1}{n(n-1)}\left(n\sum_{i=1}^{n}\sum_{j=i}^{n}\frac{1}{j}-\sum_{i=1}^{n}\sum_{j=1}^{i}\frac{j}{i}\right)=\frac{1}{n(n-1)}\left(n\cdot n-\sum_{i=1}^{n}\frac{1}{i}\cdot\frac{i(i+1)}{2}\right)=\frac{1}{n(n-1)}\left(n^2-\frac{n^2+3n}{4}\right)=\frac{3}{4}$$

Corollary 1: The orness of OWA weights in (2) is ¼ irrespective of n.

$$w_i=(1/n)\sum_{j=1,i}1/(n-j+1) \quad (2)$$

Proof:

$$orness(W)=\frac{1}{n-1}\sum_{i=1}^{n}\left((n-i)\cdot\frac{1}{n}\sum_{j=1}^{i}\frac{1}{(n-j+1)}\right)=\frac{1}{n(n-1)}\sum_{i=1}^{n}\left((n-i)\cdot\sum_{k=n-i+1}^{n}\frac{1}{k}\right)$$

$$=\frac{1}{n(n-1)}\left(\sum_{i=1}^{n}n\cdot\sum_{k=n-i+1}^{n}\frac{1}{k}-\sum_{i=1}^{n}i\cdot\sum_{k=n-i+1}^{n}\frac{1}{k}\right)=\frac{1}{n(n-1)}\left(n\cdot\sum_{i=1}^{n}\sum_{k=n-i+1}^{n}\frac{1}{k}-\sum_{i=1}^{n}\sum_{k=n-i+1}^{n}\frac{k}{i}\right)$$

$$=\frac{1}{n(n-1)}\left(n^2-\sum_{i=1}^{n}\left(\frac{1}{i}\cdot\frac{i}{2}(2n-i+1)\right)\right)=\frac{1}{4}.$$

In general, the aggregated function values, applying the formulas in (1) and (2), can be easily obtained as follows:

$f(a_1,a_2,\cdots,a_n)=(1/n)(b_1+(b_1+b_2)/2+\cdots+(b_1+\cdots+b_{n-1})/(n-1)$ $+(b_1+\cdots+b_n)/n)$ for (1) and

$f(a_1,a_2,\cdots,a_n)=(1/n)((b_1+\cdots+b_n)/n$ $+(b_2+\cdots+b_n)/(n-1)+\cdots+(b_{n-1}+b_n)/2+b_n)$ for (2).

Theorem 2: The orness of OWA weights in (3) is ⅔ irrespective of n.

$$w_i=(n-i+1)/\sum_{j=1,n}(n-j+1)=2(n-i+1)/n(n+1) \quad (3)$$

Proof:

$$orness(W)=\frac{1}{n-1}\sum_{i=1}^{n}\left((n-i)\cdot\frac{2(n+1-i)}{n(n+1)}\right)=\frac{2}{(n-1)n(n+1)}\sum_{i=1}^{n}(n-i)(n+1-i)$$

$=\frac{2}{(n-1)n(n+1)}\sum_{i=1}^{n}\left((n^2+n)-(2n+1)\cdot i+i^2\right)$. Applying an equality $\sum_{i=1}^{n}i^2=\frac{n(n+1)(2n+1)}{6}$,

$$orness(W)=\frac{2}{(n-1)n(n+1)}\left(n(n^2+n)-(2n+1)\cdot\frac{n(n+1)}{2}+\frac{n(n+1)(2n+1)}{6}\right)=\frac{2}{3}.$$

Corollary 2: The orness of OWA weights in (4) is ⅓ irrespective of n.

$$w_i=i/\sum_{j=1,n}(n-j+1)=2i/n(n+1) \quad (4)$$

Proof: $orness(W)=\frac{1}{n-1}\sum_{i=1}^{n}\left((n-i)\cdot\frac{2i}{n(n+1)}\right)=\frac{2}{(n-1)n(n+1)}\left(n\sum_{i=1}^{n}i-\sum_{i=1}^{n}i^2\right)$

$$=\frac{2}{(n-1)n(n+1)}\left(n\cdot\frac{n(n+1)}{2}-\frac{n(n+1)(2n+1)}{6}\right)=\frac{1}{3}.$$

In general, the aggregated function values, applying the formulas in (3) and (4), can be easily obtained as follows:

$f(a_1,a_2,\cdots,a_n)=\sum_{i\in I}2(n+1-i)\cdot b_i/n(n+1)$ $=2((n+1)\sum_{i\in I}b_i-\sum_{i\in I}i\cdot b_i)/n(n+1)$ for (3) and

$f(a_1,a_2,\cdots,a_n)=\sum_{i\in I}b_i(2i/n(n+1))=2(\sum_{i\in I}i\cdot b_i)/n(n+1)$ for (4).

Theorem 3: If w_is for $i\in I$ are any collection of OWA weights having the property that $w_i\geq w_j$ for $i<j$, then orness Ω belongs to the interval $0.5<\Omega\leq 1$. If w_is are any collection of OWA weights having the property that $w_i\leq w_j$ for $i<j$, then $0\leq\Omega<0.5$.

Proof: See the paper by Filev and Yager [2].

Remark 1: Let us denote by $W(k)$ the OWA weighting vector with a constant value of orness k and by $w_i(k)$ the ith element of $W(k)$. The weights W(⅔) and W(¾) belonging to the interval $0.5<\Omega\leq 1$ maintain the relation $w_i\geq w_j$ for $i<j$. The weights W(¼) and W(⅓) belonging to the interval $0\leq\Omega<0.5$ also maintain the relation $w_i\leq w_j$ for $i<j$. For the weights W(¾), w_i(¾)>w_j(¾) for $i<j$ since $(1/n)\sum_{k=i,n}(1/k)>(1/n)\sum_{k=j,n}(1/k)$ and for the weights W(⅔), w_i(⅔)>w_j(⅔) for $i<j$ since $2(n+1-i)/n(n+1)>2(n+1-j)/n(n+1)$. This can be proved for the cases of the weights W(¼) and W(⅓) in a similar way.

From the weighting functions in formulas (1) and (3), it is conceived that the weights W(¾) are steeper than the weights W(⅔) when we plot object numbers and corresponding weights in XY axis respectively. In other words, the weights W(¾) assign relatively greater weights to preceding objects. To prove this conjecture, we define $Q_k=\sum_{j=1,k}w_j$, where $Q_n=1$ and $Q_k\geq Q_{k-1}$.

Theorem 4: The weights W(¾) are steeper than the weights W(⅔). In other words, Q_k(¾)>Q_k(⅔) for $k=2,3,\cdots,n$, where Q_k(¾)$=\sum_{j=1,k}w_j$(¾) and Q_k(⅔)$=\sum_{j=1,k}w_j$(⅔).

Proof: For $k\geq 2$,

$$Q_k(3/4)=\sum_{j=1}^{k}\left(\frac{1}{n}\sum_{p=j}^{n}\frac{1}{p}\right)=\frac{1}{n}\left(\sum_{p=1}^{k}k\cdot\frac{1}{k}+k\sum_{p=k+1}^{n}\frac{1}{p}\right)>\frac{1}{n}\left(k+k\frac{n-k}{n}\right)>\frac{1}{n}\left(k+k\frac{n-k}{n+1}\right)$$

$=\frac{1}{n}\left(\frac{k+2nk-k^2}{n+1}\right)=\sum_{j=1}^{k}\frac{2(n+1-j)}{n(n+1)}=Q_k(2/3)$. Thus, Q_k(¾)>Q_k(⅔) holds.

We can also prove a relation Q_k(⅓)>Q_k(¼) for $k=2,3,\cdots,n$, analogously.

In what follows, we shall investigate the characteristics of proposed OWA weights in terms of dispersion which can be gauged by well-known entropy measure $disp(W)=-\sum_{i\in I}w_i\cdot\ln w_i$. O'Hagan [7] determines a special class of OWA operators having maximal entropy of the OWA weights (MEOWA) for some predefined value of orness. It is well-known that if a weighting vector W is optimal under some predefined value of orness α, then its reverse, denoted by W^R, and defined as $w_i^R=w_{n-i+1}$ is also optimal under degree of

<Table 1> Proposed Weighting Method when *orness* = 0.75

	Proposed weights when *ORNESS* = 0.75										
n	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	E
2	.750	.250									.562
3	.611	.278	.111								.901
4	.521	.271	.146	.063							1.148
5	.457	.257	.157	.090	.040						1.343
6	.408	.242	.158	.103	.061	.028					1.505
7	.370	.228	.156	.109	.073	.044	.020				1.644
8	.340	.215	.152	.111	.079	.054	.034	.016			1.765
9	.314	.203	.148	.111	.083	.061	.042	.026	.012		1.873
10	.293	.193	.143	.110	.085	.065	.048	.034	.021	.010	1.970

<Table 2> Weights of Newly Generated Weights using Known End Points when *orness* = 0.7

	end point (2/3, 3/4)										
n	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	E
2	.700	.300									.611
3	.544	.311	.144								.974
4	.448	.288	.178	.085							1.235
5	.383	.263	.183	.116	.056						1.441
6	.335	.240	.178	.127	.082	.040					1.610
7	.298	.220	.170	.129	.093	.061	.030				1.754
8	.269	.203	.161	.128	.098	.072	.047	.023			1.880
9	.246	.188	.152	.124	.100	.078	.057	.037	.018		.1991
10	.226	.175	.144	.120	.099	.080	.063	.046	.030	.015	2.092

orness (1- α) and $disp(W^R)=disp(W)$. It is easy to show that proposed weighting functions $W(k)$ for k=¼, ⅓, ⅔, ¾, satisfy this property as well though they are not ones derived by maximal entropy method.

Then, our basic concern is that proposed weighting functions $W(k)$ for k=¼, ⅓, ⅔, ¾ lie in what level of dispersion, compared to the weights generated by maximal entropy method. This consideration can be set forth by each comparison of MEOWA weights and the weights by proposed method for a predefined value of orness. In Table 1, the weights generated by MEOWA and proposed method are listed when orness is set at 0.75. It is obvious that only small differences between them exist at third decimal places of the numbers. This holds true for MEOWA weights and proposed weights when orness is fixed at 0.25, 0.667, and 0.334 respectively. The other concern we want to address falls into a case that decision maker wants to make an aggregation of objects at some other level of orness except four specific levels of orness. To deal with this situation, a way of determining OWA weights is described below.

Theorem 5: A new OWA weighting vector having a predefined and a constant value of orness can be generated by a convex combination of any two OWA weighting functions already known to have constant level of orness.

Proof: If a predefined value of orness is k ($0<k<1$), let us denote a newly generated OWA weighting vector as $W^{New}(k)$. The weights $W^{New}(k)$ can be generated by a convex combination of $W(k')$ and $W(k'')$ which are the OWA weighting functions having constant levels of orness k' and k'' respectively. Then the ith element of $W^{New}(k)$ becomes $w_i^{New}= \beta w_i(k')+(1-\beta)w_i(k'')$ for $i\in I$ and $\beta\in[0, 1]$. We can always find out $\beta\in[0, 1]$ satisfying $k=\beta\cdot k'+(1-\beta)\cdot k''$. Then, applying the new weights $W^{New}(k)$ into orness measure, we obtain OWA weights with a predefined and a constant value of orness k.

$orness(W^{New})=(\sum_{i\in I}(n-i)w_i^{New})/(n-1)=\sum_{i\in I}(n-i)(\beta w_i(k')+(1-\beta)w_i(k''))/(n-1)$
$=\beta(\sum_{i\in I}(n-i)w_i(k'))/(n-1)+(1-\beta)(\sum_{i\in I}(n-i)w_i(k''))/(n-1)=\beta\cdot k'+(1-\beta)\cdot k''=k.$

Example. Suppose that we want to generate OWA weights with orness 0.7 from the OWA weighting vectors W(⅔) and W(¾), then we simply solve the equation $(2/3)\beta+(3/4)(1-\beta)=0.7$, which results in $\beta=0.6$. Thus if $n=3$, newly generated OWA weighting vector with *orness*=0.7 becomes

$W^{New}(0.7)=((0.6)(0.5)+(0.4)(0.611), (0.6)(0.333)+(0.4)(0.278), (0.6)(0.167)+(0.4)(0.111))$
$= (0.544, 0.311, 0.145).$

The orness of the newly generated weights is, of course, $orness(W^{New}(0.7))=0.7$.

Corollary 3: A convex combination of the weights $W(⅔)$ and $W(⅓)$ with β=0.5 results in well-known OWA average operator, $W^{New}(½)=(1/n,1/n,\cdots,1/n)$.

Proof:

$w_i^{New}(½)=(1/2)w_i(⅔)+(1/2)w_i(⅓)=2(n+1-i)/2n(n+1)+2i/2n(n+1)=1/n$.

The new OWA operator weights with a predefined value of orness can be constructed by using known end points of orness which encompass the predefined value of orness. The difficult is, however, that there exist many alternatives to be chosen for end points. If we want to generate new OWA operator weights with e.g., *orness*=0.7, then we can make not a few end points which all make the weights locate at *orness*=0.7 by varying parameter β. For instance, the pairs of end points such as (⅔, ¾), (½, ¾), (⅓, ¾), and (¼, ¾) can be chosen as one of the options, considering β=0.6, 0.2, 0.12 and 0.1 respectively. Our consideration is that which of them is the most appropriate to use in the aggregation. As one of criteria to be considered, let us suppose that we want to select newly generated weights that result in a maximum entropy. In Table 2, we present newly generated weights by convex combinations of end points (⅔, ¾) at *orness*=0.7. In this case, only small differences can be found at third decimal places of the numbers in all comparisons. The OWA weights generated at a predefined value of orness are expected to make almost no numerical difference with MEOWA weights in terms of dispersion since parameter β varying to maintain a predefined value of orness smoothes the differences between them and thus finally make their entropies close to each other.

3. Concluding remarks

We present analytic forms of OWA weighting functions of which orness is constant irrespective of the number of objectives. Further, using the analytic weighting functions and well-known OWA operators including max, min, and average, any new OWA weights of which orness is also constant can be generated on the orness scale.

Furthermore the values obtained via the formulas are numerically close to those obtained by the MEOWA. Thus in a situation where a priori degree of optimism is specified by decision maker, we can simply apply the analytic weighting functions or generate OWA weights that have a predefined value of orness to perform aggregation of multiple objects.

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