On the Intuitionistic Fuzzy Metric Spaces

직관적 퍼지거리공간에 관하여

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Abstract

In this paper, we define precompact set in intuitionistic fuzzymetric spaces and prove that any subset of an intuitionistic fuzzy metric space is compact if and only if it is precompact and complete. Also we define topologically complete intuitionistic fuzzy metrizable spaces and prove that any G_δ set in a complete intuitionistic fuzzy metric spaces is a topologically complete intuitionistic fuzzy metrizable space and vice versa.

Key Words: Intuitionistic fuzzy metric spaces, fuzzy metric spaces, precompact, continuous t-norm, compact, intuitionistic fuzzy normed spaces.

1. Introduction and Preliminaries

The theory of fuzzy sets was introduced by L. Zadeh in 1965 [24]. After the pioneering work of Zadeh, there has been a great effort to obtain fuzzy analogues of classical theories. Among other fields, a progressive developments is made in the field of fuzzy topology. The concept of fuzzy topology may have very important applications in quantum particle physics particularly in connections with both string and ε^∞ theory which were given and studied by Elnaschie [8,9]. One of the most important problems in fuzzy topology is to obtain an appropriate concept of intuitionistic fuzzy metric space. This problem has been investigated by J.H. Park [21]. He has introduced and studied a notion of intuitionistic fuzzy metric space. We recall it.

Definition 1.1 The 5-tuple $(X, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2\times(0,\infty)$ satisfying the following conditions: for all $x, y, z \in X$, $s, t > 0$

(a) $M(x,y,t)+N(x,y,t)\leq 1$;
(b) $M(x,y,t) > 0$;
(c) $M(x,y,t)=1$ if and only if $x=y$;
(d) $M(x,y,t)=M(y,x,t)$;
(e) $M(x,y,t)*M(y,z,s)\leq M(x,z,t+s)$;
(f) $M(x,y,\cdot): (0,\infty)\to(0,1]$ is continuous;
(g) $N(x,y,t) > 0$;
(h) $N(x,y,t)=0$ if and only if $x=y$;
(i) $N(x,y,t) = N(y,x,t)$;
(j) $N(x,y,t) \diamond N(y,z,s) \geq N(x,z,t+s)$;

(k) $N(x, y, \cdot): (0, \infty) \to (0, 1]$ is continuous.

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Example 1.2 Let (X, d) be a metric space. Denote $a*b = ab$ and $a \diamond b = \min\{1, a+b\}$ for all $a, b \in [0, 1]$ and let M_d and N_d be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$M_d(x, y, t) = \frac{h t^n}{h t^n + m d(x, y)}, \text{ and}$$
$$N_d(x, y, t) = \frac{d(x, y)}{k t^n + m\, d(x, y)}$$

for all $h, k, m, n \in R^+$. Then $(X, M_d, N_d, *, \diamond)$ is an intuitionistic fuzzy metric space.

Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space, For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by
$B(x, r, t) = \{y \in X: M(x, y, t) > 1 - r,\ N(x, y, t) < r\}$.

Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. Let $\tau_{(M,N)}$ be the set of all $A \subset X$ with $x \in A$ if and only if there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Then $\tau_{(M,N)}$ is a topology on X (induced by the intuitionistic fuzzy metric (M, N)). This topology is Hausdorff and first countable. A sequence $\{x_n\}$ in X converges to x if and only if $M(x_n, x, t) \to 1$ and $N(x_n, x, t) \to 0$ as $n \to \infty$, for each $t > 0$. It is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in$ N such that $M(x_n, x_m, t) > 1 - \varepsilon$ and $N(x_n, x_m, t) < \varepsilon$ for each $n, m \geq n_0$. The intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be complete if every Cauchy sequence is convergent. A subset A of X is said to be IF-bounded if there exist $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$ and $N(x, y, t) < r$ for all $x, y \in A$.

A collection U of open sets is called an open cover of A if $A \subset \bigcup\{U: U \in \text{U}\}$. A subspace A of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is compact if every open cover of A has a finite subcover. If every sequence in A has a convergent subsequence to a point in A, then it is called sequential compact.

Theorem 1.3 [15] In an intuitionistic fuzzy metric space every compact set is closed and IF-bounded.

Theorem 1.4 [15] Every closed subset of a complete intuitionistic fuzzy metric space is complete.

2. Precompact Intuitionistic Fuzzy Metric Spaces

Definition 2.1 Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space and $A \subset X$. We say A is precompact if for each $0 < r < 1$ and $t > 0$ there exists a finite subset S of A such that

$$A \subseteq \bigcup_{x \in S} B(x, r, t).$$

Lemma 2.2 Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space and $A \subset X$. Then A is a precompact set if and only if for every $0 < r < 1$ and $t > 0$, there exists a finite subset S of X such that
$A \subseteq \bigcup_{x \in S} B(x, r, t)$.

Lemma 2.3 Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space and $A \subset X$. If A is a precompact set, then so is its closure $\overline{A}$.

Theorem 2.4 Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space and $A \subset X$. Then A is a precompact set if and only if every sequence has a Cauchy subsequence.

Lemma 2.5 Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. If a Cauchy sequence clusters to a point $x \in X$, then the sequence converges to x.

Lemma 2.6 Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. Then $(X, \tau_{(M,N)})$ is a metrizable topological space.

Note that, in every metrizable space every sequentially compact set is compact.

Corollary 2.6 A subset A of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is compact if and only if it is precompact and complete.

3. Complete Intuitionistic Fuzzy Metric Spaces

Lemma 3.1 Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space and let $\lambda, \eta \in (0,1)$ such that $\lambda + \eta \leq 1$. Then there exists an intuitionistic fuzzy metric (m, n) on X such that $m(x, y, t) \geq \lambda$ and $n(x, y, t) \leq \eta$ for each $x, y \in X$ and $t > 0$ and (m, n) and (M, N) induce the same topology on X.

The intuitionistic fuzzy metric (m, n) in above lemma is said to be bounded by (λ, η).

Definition 3.2 Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space, $x \in X$ and $\phi \neq A \subseteq X$. We define

$$D(x, A, t) = \sup\{M(x, y, t) : y \in A\} \quad (t > 0),$$
$$C(x, A, t) = \inf\{N(x, y, t) : y \in A\} \quad (t > 0).$$

Note that $D(x, A, t)$ and $C(x, A, t)$ are a degree of closeness and a degree of non-closeness of x to A at t, respectively.

Definition 3.3 A topological space is called a topologically complete intuitionistic fuzzy metrizable space if there exists a complete intuitionistic fuzzy metric inducing the given topology on it.

Example 3.4 Let $X = (0, 1]$. The intuitionistic fuzzy metric space $(X, M, N, \min, \max)$ where $M(x, y, t) = \frac{t}{t + |x - y|}$ and $N(x, y, t) = \frac{|x - y|}{t + |x - y|}$ (standard intuitionistic fuzzy metric, see [15]) is not complete, because the Cauchy sequence $\{\frac{1}{n}\}$ in this space is not convergent. Now consider the 5-tuple $(X, m, n, \min, \max)$, where

$$m(x, y, t) = \frac{t}{t + |x - y| + |\frac{1}{x} - \frac{1}{y}|} \text{ and}$$

$$n(x, y, t) = \frac{|x - y| + |\frac{1}{x} - \frac{1}{y}|}{t + |x - y| + |\frac{1}{x} - \frac{1}{y}|}.$$

It is easy to show that $(X, m, n, \min, \max)$ is an intuitionistic fuzzy metric space which is complete. Since, x_n tends to x with respect to intuitionistic fuzzy metric (M, N), if and only if $|x_n - x| \to 0$, if and only if x_n tends to x with respect to intuitionistic fuzzy metric (m, n), hence (M, N) and (m, n) are equivalent intuitionistic fuzzy metrics. Hence the intuitionistic fuzzy metric space $(X, M, N, \min, \max)$ is topologically complete intuitionistic fuzzy metrizable.

Lemma 3.5 Intuitionistic fuzzy metrizability is preserved under countable Cartesian product.

Theorem 3.6 An open subspace of a complete intuitionistic fuzzy metrizable space is a topologically complete intuitionistic fuzzy metrizable space.

Corollary 3.7 A G_δ set in a complete intuitionistic fuzzy metric space is a topologically complete intuitionistic fuzzy metrizable space.

Theorem 3.8 Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space and X be a topologically complete intuitionistic fuzzy metrizable subspace of Y. Then X is a G_δ subset of Y.

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