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On compact set-valued Choquet integrals

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Abstract

We note that Jang et al. studied closed set-valued Choquet integrals with respect to fuzzy measures. In this paper, we consider Choquet integrals of compact set-valued functions, and prove some properties of them. In particular, using compact set-valued functions, instead of interval valued we investigate characterization of compact set-valued Choquet integrals.

1. Introduction

In this paper, we consider Choquet integrals of compact set-valued functions. We note that Jang et al. [1] studied closed set-valued Choquet integrals with respect to fuzzy measures. In Section 2, we define Choquet integrals of compact set-valued functions and discuss their basis properties. In Section 3, using these definitions and properties, we investigate characterization of compact set-valued Choquet integrals.

2. Preliminaries and definitions

Throughout this paper, we assume that X is a locally compact Hausdorff space, K is the class of continuous functions on X with compact support, B is the class of Borel sets, C is the class of compact sets, and O is the class open set. The class of measurable functions is denoted by M and the class of non-negative measurable functions is denoted by M^+.

A non-additive measure on a measurable space (X, B) is an extended real-valued function $\mu : B \to [0, \infty]$ satisfying

(1) $\mu(\varnothing) = 0$,

(2) $\mu(A) \le \mu(B)$,

whenever $A, B \in B, A \in B$

When $\mu(X) < \infty$, we define the conjugate μ^C of μ by

$$\mu^C(A) = \mu(X) - \mu(A^C),$$

where A^C is thee complement of $A \in B$.

Recall that a function $f: X \to [0, \infty]$ is said to be measurable if $\{x | f(x) > \alpha\} \in B$ for all $\alpha \in (-\infty, \infty)$. [1,2,3]

Definition 2.1 [4,5,7] (1) The Choquet integral of measurable function $f \in M^+$ with respect to a fuzzy measure μ is defined by

$$(C)\int f\, d\mu = \int_0^\infty \mu_f(r) dr,$$

where the integral on the right-hand side is an ordinary one.

(2) A measurable function f is called integrable if the Choquet of f can be defined and its value is finite.

Throughout this paper, R^+ will denote the interval $[0, \infty)$, $I(R^+) = \{[a, b] | a, b \in R^+$ and $a \le b\}$. Then an element in $I(R^+)$ is called an interval number. On the interval number set, we define; for each pair $[a, b], [c, d] \in I(R^+)$ and $k \in R^+$,

$$[a, b] + [c, d] = [a + c, b + d],$$
$$[a, b] \cdot [c, d] = [a \cdot c, b \cdot d],$$
$$k[a, b] = [ka, kb],$$

$[a, b] \le [c, d]$ if and only if $a \le c$ and $b \le d$. Then $(I(R^+), d_H)$ is a metric space, where of the Hausdorff metric defined by

$$d_H(A, B) = max\{sup_{x \in A} inf_{y \in B} | x - y|, \; sup_{y \in B} inf_{x \in A} | x - y| \}$$

for all $A, B \in I(R^+)$. By the definition of the Hausdorff metric, we have immediately the following proposition.

Proposition 2.2 [4,5,7] For each pair $[a, b]$, $[c, d] \in I(R^+)$,

$$d_H([a, b], [c, d]) = max\{| a - c|, | b - d| \}.$$

Let $C(R^+)$ be the class of compact subsets of R^+. Throughout this paper, we consider a compact set-valued function $F: X \to C(R^+) \setminus \{\varnothing\}$ and an interval number-valued function $F: X \to I(R^+) \setminus \{\varnothing\}$. We denote that

$d_H - \lim_{n\to\infty} A_n = A$ if and only if $\lim_{n\to\infty} d_H(A_n, A) = 0$, where $A \in I(R^+)$ and $\{A_n\} \subset I(R^+)$.

Definition 2.3 A compact set-valued function F is said to be measurable if for each open set $O \subset R^+$,

$$F^{-1}(O) = \{x | F(x) \cap O\} \ne \varnothing \in B.$$

Definition 2.4 Let F be a compact set-valued function. A measurable function $f: X \to R^+$ satisfying

$$f(x) \in F(x), \forall x \in X$$

is called a measurable selection of F.

We say $f: X \to R^+$ is in $L_1(\mu)$ if and only if f is measurable and $(C)\int f\, d\mu < \infty$.

We note that "$x \in X \;\; \mu - a.e.$" stand for "$x \in X \;\; \mu$ −almost everywhere". The property $P(x)$ holds for $x \in X \;\; \mu - a.e.$ means that there is a measurable set A such that $\mu(A) = 0$ and the property $P(x)$ holds for all $x \in A^C$, where A^C is the complement of A.

Definition 2.5 Let F be a compact set-valued function, let μ be a non-additive measure and $A \in B$:

(1) The Choquet integral of F on A is defined by $(C)\int F d\mu = \{(C)\int_A f d\mu | f \in S(F)\}$,

where $S(F)$ is the family of $\mu - a.e.$ measurable selections of F, that is,

$S(F) = \{f \in M^+ | f(x) \in F(x), x \in X, \mu - a.e\}$.

(2) A compact set-valued function F is said to be Choquet integrable if $(C)\int F\, d\mu \ne \varnothing$, and it is said to be Choquet integrable if $(C)\int F d\mu$ exists and dose not include ∞.

(3) A compact set-valued function F is said to be Choquet integrably bounded if there is a function $g \in M^+$ such that

$F(x) = sup_{r \in F(x)} |r| \leq g(x), \ \forall x \in X.$

3. Main results

In this section, we consider the following classes of closed set-valued functions and compact set-valued functions;

$\mathbb{T} = \{F | F : X \to C(R^+)$ is measurable closed set-valued function and Choquet integrably bounded $\}$

$\mathbb{T}_1 = \{F | F : X \to C(R^+)$ is measurable compact set-valued function and Choquet integrably bounded $\}$

Let A be a set and $cl(A)$ means the closure of A in R^+.

Theorem 3.1 If $F \in \mathbb{T}_1$, then $(C)\int F d\mu$ is compact.

Proof. We assume that F is measurable compact set-valued function on X and shall show that $(C)\int F d\mu$ is closed and bounded on $C(R^+) \setminus \{\emptyset\}$. Let F be closed. Then we have $(C)\int_A F d\mu = \{(C)\int_A f d\mu | f \in S(F)\}$ is closed. Hence, Let F be Choquet integrably bounded. Then there is $g(x)$ such that $F(x) \leq g(x)$. Thus we obtain $F(x) \subset [0, g(x)]$.

For every $f \in S(F)$,

$$0 \leq (C)\int f \, d\mu \leq (C)\int g \, d\mu. \tag{3.1}$$

Since $f^- \leq |f|$, we have

$$0 \leq (C)\int f^- d\mu \leq (C)\int g d\mu. \tag{3.2}$$

Since $f^+ \leq |f| = f^+ + f^-$, we have

$$0 \leq (C)\int f^+ d\mu \leq (C)\int g d\mu. \tag{3.3}$$

By (3.1),(3.2) and (3.3),

$$(C)\int f \, d\mu \leq (C)\int g d\mu$$

for every $f \in S(F)$ such that $f(x) \leq g(x)$. So $(C)\int F d\mu$ is bounded. Thus $(C)\int F d\mu$ is compact. □

Corollary 3.2 If $F \in \mathbb{T}$ and $G \in \mathbb{T}_1$, then $(C)\int (F \cap G) \, d\mu$ is compact.

Definition 3.3 Let $G, H \in \mathbb{T}$. We define $(G \cup H)(x) = G(x) \cup H(x)$ for all $x \in X$.

Theorem 3.4 (1) If $F, G \in \mathbb{T}$, then $F \cup G \in \mathbb{T}$ and $F \cap G \in \mathbb{T}$.
(2) If $F, G \in \mathbb{T}_1$, then $F \cup G \in \mathbb{T}_1$ and $F \cap G \in \mathbb{T}_1$.

Theorem 3.5 If $F, G \in \mathbb{T}_1$, then

$$(C)\int (F \cup G) \, d\mu = (C)\int F \, d\mu \cup (C)\int G \, d\mu.$$

Theorem 3.6 If $F, G \in \mathbb{T}_1$, then

$$(C)\int (F \cap G) \, d\mu \subset (C)\int F \, d\mu \cap (C)\int G \, d\mu.$$

Definition 3.7 Let $G, H \in \mathbb{T}$. We define

$$G \uplus H = cl(G \cup H).$$

Theorem 3.8 If $F, G \in \mathbb{T}_1$, then

$$(C)\int (F \uplus G) \, d\mu = (C)\int F \, d\mu \uplus (C)\int G \, d\mu.$$

Proof. By Theorem 3.1, $cl\{(C)\int F d\mu\} = (C)\int F d\mu$. If F and G are measurable compact set-valued functions on X, by Theorem 3.5 , we have

$$(C)\int (F \uplus G)\, d\mu = (C)\int cl(F \cup G)\, d\mu$$

$$= (C)[\int (cl(F) \cup cl(G))\, d\mu]$$

$$= (C)\int cl(F)\, d\mu \ \cup \ (C)\int cl(G)\, d\mu$$

$$= cl\,(\,(C)\int F\, d\mu \ \cup \ (C)\int G\, d\mu\,)$$

$$= (C)\int F\, d\mu \uplus (C)\int G\, d\mu. \qquad \square$$

Corollary 3.9 Let $F_k,\ k = 1, 2, 3, \cdots$. Then we have

$$(C)\int \uplus_{k=1}^{n} F_k\, d\mu = \uplus_{k=1}^{n} (C)\int F_k\, d\mu\,.$$

We define $aA = \{ax \mid x \in A\}$, $A + B = \{x + y \mid x \in A, y \in B\}$, $A,\ B \in C(R^+)$ and $a \in R^+$.

Definition 3.10 Let $G, H \in \mathbb{T}$. We define

$$G \oplus H = cl(G + H).$$

Theorem 3.11 If $F, G \in \mathbb{T}_1$, then

$$(C)\int (F \oplus G)\, d\mu = (C)\int F\, d\mu \oplus (C)\int G\, d\mu.$$

4. References

[1] J. Aubin, "Set-valued analysis", Birkauser Boston, 1990.

[2] R .J. Aumann, "Integrals of set-valued functions", J. Math. Anal. Appl. Vol. 12, pp 1-12, 1965.

[3] F. Hiai and H. Umegaki, "Integrals, conditional expectations, and martingales of multivalued functions", J. Multi. Analysis Vol. 7, pp 149-182, 1977.

[4] Lee-Chae Jang. T. Kim, and Jong Duek Jeon, "On comonotonically additive interval-valued Choquet integrals(II)", KFIS, Vol. 14, No. 1, pp.33-38, 2004.

[5] Lee-Chae Jang. T. Kim, and Jong Duek Jeon, "On set-valued Choquet integrals and convergence theorem (II) ", Bull. Kerean Math.Soc., Vol. 20, No. 1, pp.139-147, 2003.

[6] Hyun Mee Kim, Lee-Chae Jang and T. Kim, "On closed set-valued Choquet integrals with respect to non-monotonic fuzzy measures", Proceedings of the Jangjeon Mathematical Society,Vol.7, pp 109-116, 2004.

[7] D. Zhang and Z. Wang, "Fuzzy integrals of fuzzy-valued functions", Fuzzy sets and systems, Vol. 54, pp 63-67, 1993.