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Laplace Transforms of First Exit Times for Compound Poisson Dams

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Abstract

An infinite dam with compound Poisson inputs and a state-dependent release rate is considered. We build the Kolmogorov's backward differential equation and solve it to obtain the Laplace transforms of the first exit times for this dam.

Keywords: first exit times, compound Poisson dams, Laplace transforms

1. Introduction

We consider a dam of infinite capacity which water flows in according to a compound Poisson process $\{A(t), t \geq 0\}$ and flows out with a release rate $r(x)$ depending on the present content x of the dam. The compound Poisson process $\{A(t), t \geq 0\}$ can be expressed by

$$A(t) = \sum_{n=1}^{N(t)} S_n,$$

where $\{N(t), t \geq 0\}$ is a Poisson process with intensity λ and $S_1, S_2, \cdots$ are i.i.d. random variables with common distribution G. We assume that $G(0)=0$ and S_n are independent of the arrival process $\{N(t), t \geq 0\}$. If we let $X(t)$ be the content at time t, then its sample paths satisfy the following storage equation:

$$X(t) = x + A(t) - \int_0^t r(X(s))\, ds, \quad t \geq 0$$

with $X(0)=x$ being the initial content of the dam. We assume that $r(0)=0$ and $r(\cdot)$ is strictly positive, left continuous, and has a strictly positive right limit at every point in $(0, \infty)$. We also assume that for any $0 < x < \infty$,

$$\int_0^x \frac{1}{r(y)}\, dy < \infty,$$

meaning that level zero can be reached in a finite amount of time from any level $x > 0$.

Harrison and Resnick(1976) derived the first exit probability that the content

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exceeds one specified level before it reaches the other specified level, starting from some level between them. For the case of constant release rate, Perry et al.(2002) found the Laplace transform of the first exit times for the infinite dam while Bae et al.(2001) obtained the Laplace transform of the wet period of the finite dam. Kinateder and Lee(2000) determined explicitly the expectation of the first exit time by adopting the martingale arguments in Rosenkrantz(1983) when the release rate is constant. Kella and Stadje(2001) especially computed the Laplace transform of the first hitting times for the dam with exponential jumps and linear release rate.

In this paper, we extend the results in Bae et al.(2001) of constant release rate into the case of the general release rate function. In the following section, we obtain the Laplace transform of the first exit times for the infinite dam. To do so, we derive an integro-differential equation based on the method of Kolmogorov's backward differential equation and solve it in terms of a certain positive kernel.

2. Laplace transforms of the first exit times

We assume that $0 \le \alpha < \beta < \infty$. For $\alpha \le x \le \beta$, we define

$$T_{\alpha,\beta}(x) = inf\,\{t \ge 0 | X(t) \notin (\alpha, \beta],\ X(0) = x\}$$

representing the first exit time from $(\alpha, \beta]$ for the process $\{X(t), t \ge 0\}$ starting at $X(0) = x$. Then, the Laplace transform of $T_{\alpha,\beta}(x)$ can be given by

$$\begin{aligned} \phi_{\alpha,\beta}(x, \theta) :&= E[e^{-\theta T_{\alpha,\beta}(x)}] \\ &= \phi^1_{\alpha,\beta}(x, \theta) + \phi^2_{\alpha,\beta}(x, \theta), \quad \theta \ge 0, \end{aligned}$$

where

$$\phi^1_{\alpha,\beta}(x, \theta) := E[e^{-\theta T_{\alpha,\beta}(x)} 1_{\{X(T_{\alpha,\beta}(x)) > \beta\}}] \tag{1}$$

and

$$\phi^2_{\alpha,\beta}(x, \theta) := E[e^{-\theta T_{\alpha,\beta}(x)} 1_{\{X(T_{\alpha,\beta}(x)) = \alpha\}}]. \tag{2}$$

Here if we define

$$T^1_{\alpha,\beta}(x) = \begin{cases} \infty & if\ \ X(T_{\alpha,\beta}(x)) = \alpha, \\ T_{\alpha,\beta}(x) & if\ \ X(T_{\alpha,\beta}(x)) > \beta, \end{cases}$$

and

$$T^2_{\alpha,\beta}(x) = \begin{cases} T_{\alpha,\beta}(x) & if\ \ X(T_{\alpha,\beta}(x)) = \alpha, \\ \infty & if\ \ X(T_{\alpha,\beta}(x)) > \beta, \end{cases}$$

then the functions $\phi^1_{\alpha,\beta}$ and $\phi^2_{\alpha,\beta}$ of (1) and (2) are the Laplace transforms of $T^1_{\alpha,\beta}(x)$ and $T^2_{\alpha,\beta}(x)$, respectively, given by

$$\phi^1_{\alpha,\beta}(x, \theta) = E[e^{-\theta\, T^1_{\alpha,\beta}(x)}], \quad \theta > 0$$

and

$$\phi^2{}_{\alpha,\beta}(x,\theta)=E[e^{-\theta T^2{}_{\alpha,\beta}(x)}],\quad \theta>0.$$

Let $F^1{}_{\alpha,\beta}(x,t)=P\{T^1{}_{\alpha,\beta}(x)\le t\}$. Clearly $T^1{}_{\alpha,\beta}(\alpha)=\infty$ and then $F^1{}_{\alpha,\beta}(\alpha,t)=0$ for all $t\ge 0$. For $\alpha < x\le \beta$, we employ the method of Kolmogorov's backward differential equation to get $F^1{}_{\alpha,\beta}(x,t)$. Conditioning on whether an input of water into the dam occurs or not during the infinitesimal interval $(0,\Delta t]$ gives

$$T^1{}_{\alpha,\beta}(x)=\begin{cases} T^1{}_{\alpha,\beta}(x-r(x)\Delta t)+\Delta t, & \text{if no inputs occur,}\\ T^1{}_{\alpha,\beta}(x-r(x)\Delta t+S)+\Delta t, & \text{if an input occurs and } S\le\beta-x+r(x)\Delta t\\ \Delta t, & \text{if an input occurs and } S>\beta-x+r(x)\Delta t\end{cases}$$

where S is the generic random variable with distribution G Since the probability that two or more inputs occur during $(0,\Delta t]$, we have that, for $t>0$,

$$\begin{aligned}F^1{}_{\alpha,\beta}(x,t)= &\ (1-\lambda\Delta t)\,F^1{}_{\alpha,\beta}(x-r(x)\Delta t,t-\Delta t)\\ &+\lambda\Delta t\int_0^{\beta-x+r(x)\Delta t}F^1{}_{\alpha,\beta}(x-r(x)\Delta t+z,t-\Delta t)dG(z)\\ &+\lambda\Delta t[1-G(\beta-x+r(x)\Delta t)]+o(\Delta t),\end{aligned}$$

where $o(\Delta t)/\Delta t$ goes to zero as $\Delta t\to 0$. Dividing each side of the above equation by Δt and letting Δt go to zero yield

$$\begin{aligned}r(x)\frac{\partial}{\partial x}F^1{}_{\alpha,\beta}(x,t)+\frac{\partial}{\partial t}F^1{}_{\alpha,\beta}(x,t)= &-\lambda F^1{}_{\alpha,\beta}(x,t)+\lambda\int_0^{\beta-x}F^1{}_{\alpha,\beta}(x+z,t)dG(z)\\ &+\lambda[1-G(\beta-x)].\end{aligned}\tag{3}$$

Putting $y=\beta-x$, $\widetilde{F^1}{}_{\alpha,\beta}(y,t)=F^1{}_{\alpha,\beta}(\beta-y,t)$ and $\widetilde{r}(y)=r(\beta-y)$ for the convenience of analysis, the equation (3) can be rewritten in

$$\begin{aligned}-\widetilde{r}(y)\frac{\partial}{\partial y}\widetilde{F^1}{}_{\alpha,\beta}(y,t)+\frac{\partial}{\partial t}\widetilde{F^1}{}_{\alpha,\beta}(y,t)= &-\lambda\widetilde{F^1}{}_{\alpha,\beta}(y,t)+\lambda\int_0^y\widetilde{F^1}{}_{\alpha,\beta}(y-z,t)dG(z)\\ &+\lambda(1-G(y)),\quad 0\le y<\beta-\alpha\end{aligned}\tag{4}$$

and the Laplace transform $\phi^1{}_{\alpha,\beta}(x,\theta)$ of $T^1{}_{\alpha,\beta}(x)$ can also be written by

$$\begin{aligned}\phi^1{}_{\alpha,\beta}(x,\theta)= &\int_0^\infty e^{-\theta t}d_tF^1{}_{\alpha,\beta}(x,t)\\ = &\int_0^\infty e^{-\theta t}d_t\widetilde{F^1}{}_{\alpha,\beta}(y,t)|_{y=\beta-x}\\ := &\ \widetilde{\phi^1}{}_{\alpha,\beta}(y,\theta).\end{aligned}$$

Since $\widetilde{\phi^1}{}_{\alpha,\beta}(y,\theta)=\theta\int_0^\infty e^{-\theta t}\widetilde{F^1}{}_{\alpha,\beta}(y,t)dt$, by multiplying $\theta e^{-\theta t}$ on both sides of the equation (4) and integrating both sides with respect to t from 0 to ∞, we have the following integro-differential equation for $\widetilde{\phi^1}{}_{\alpha,\beta}$:

$$\frac{\partial}{\partial y}\widetilde{\phi^1}{}_{\alpha,\beta}(y,\theta)=f_1(y,\theta)+\int_0^yK(y,z,\theta)d_z\widetilde{\phi^1}{}_{\alpha,\beta}(z,\theta),\tag{5}$$

where

$$f_1(y,\theta) := \widetilde{\phi^1}_{\alpha,\beta}(0,\theta)[\theta+\lambda(1-G(y))]-\lambda(1-G(y))\widetilde{\gamma}(y) \tag{6}$$

and

$$K(y,z,\theta) := \frac{\theta+\lambda[1-G(y-z)]}{\widetilde{\gamma}(y)},$$

where parameters α and β are omitted in the notations of functions $f_1(\cdot)$ and $K(\cdot)$ for the simplicity.

In a manner analogous to that in Harrison and Resnick(1976), let

$$K^1(y,z,\theta)=K(y,z,\theta), \quad 0\le z< y< \beta-\alpha, \ \ \theta\ge 0$$

and define its iterates recursively by

$$\begin{aligned} K^{n+1}(y,z,\theta) &= \int_z^y K^n(y,w,\theta)K^1(w,z,\theta)dw \\ &= \int_z^y K^1(y,w,\theta)K^n(w,z,\theta)dw, \quad 0\le z< y< \beta-\alpha, \quad \theta\ge 0, \end{aligned}$$

for $n\ge 1$. Using the bound $K^1(y,z,\theta)\le(\theta+\lambda)/\widetilde{\gamma}(y)$, it follows easily by induction that

$$K^{n+1}(y,z,\theta)\le \frac{(\theta+\lambda)^{n+1}[\int_z^y \frac{1}{\widetilde{\gamma}(w)}dw]^n}{\widetilde{\gamma}(y)n!}, \quad 0\le z< y< \beta-\alpha, \quad \theta\ge 0,$$

for all $n\ge 1$. Thus the kernel

$$K^*(y,z,\theta) := \sum_{n=1}^{\infty} K^n(y,z,\theta)$$

is well-defined. Iterating the relation (5) for $N-1$ times gives

$$\frac{\partial}{\partial y}\widetilde{\phi^1_{\alpha,\beta}}(y,\theta) = f_1(y,\theta)+\int_0^y f_1(z,\theta)\sum_{n=1}^{N-1}K^n(y,z,\theta)dz+\int_0^y K^N(y,z,\theta)d_z\widetilde{\phi^1_{\alpha,\beta}}(z,\theta).$$

Letting $N\to\infty$ and using the dominated convergence theorem, we conclude that the derivative $\partial\,\widetilde{\phi^1}_{\alpha,\beta}(y,\theta)/\partial y$ is given by

$$\frac{\partial}{\partial y}\widetilde{\phi^1_{\alpha,\beta}}(y,\theta) = f_1(y,\theta)+\int_0^y f_1(z,\theta)K^*(y,z,\theta)dz.$$

Substituting $f_1(y,\theta)$ of (6) into the above equation and then integrating it with respect to y from 0 to z yield

$$\begin{aligned} \widetilde{\phi^1}_{\alpha,\beta}(z,\theta) &= \widetilde{\phi^1}_{\alpha,\beta}(0,\theta)[1+\int_0^z K^*(y,0,\theta)dy] \\ &\quad -\int_0^z K(w,0,0)[1+\int_w^z K^*(y,w,\theta)dy]dw, \quad 0\le z< \beta-\alpha, \ \ \theta\ge 0. \end{aligned} \tag{7}$$

From the boundary condition that it follows that

$$\widetilde{\phi}^1_{\alpha,\beta}(0,\theta)=\frac{\int_0^{\beta-\alpha}K(w,0,0)[1+\int_w^{\beta-\alpha}K^*(y,w,\theta)dy]dw}{1+\int_0^{\beta-\alpha}K^*(y,0,\theta)dy}$$

which completes the function $\widetilde{\phi}^1_{\alpha,\beta}(z,\theta)$ of (7).

By the similar method we can show that the Laplace transform $\widetilde{\phi}^2_{\alpha,\beta}(z,\theta):=\phi^2_{\alpha,\beta}(\beta-z,\theta)$ is given by

$$\widetilde{\phi}^2_{\alpha,\beta}(z,\theta)=\frac{1+\int_0^{z}K^*(y,0,\theta)dy}{1+\int_0^{\beta-\alpha}K^*(y,0,\theta)dy},\quad 0\le z<\beta-\alpha,\quad \theta\ge 0.$$

Remark We note that $\phi^1_{\alpha,\beta}(x,0)$ is the probability that the content process of the dam operating with the release rate function $r(\cdot)$, which starts from x, up-crosses level β without reaching level α and is given by

$$\phi^1_{\alpha,\beta}(x,0)=\int_\alpha^x K^*(\beta-y,0,0)dy\,1+\int_\alpha^\beta K^*(\beta-y,0,0)dy,$$

which coincides the first exit probability $U(x)$ for the case that $a=\alpha$ and $b=\beta$ in Harrison and Resnick(1976). Moreover the probability $\phi^2_{\alpha,\beta}(x,0)$ is that the dam process reaches α before hitting β when it starts from x, that is, $\phi^2_{\alpha,\beta}(x,0)=1-\phi^1_{\alpha,\beta}(x,0)$.

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