

Optimal Periodic Preventive Maintenance Policies Based on Two Imperfect PM Models

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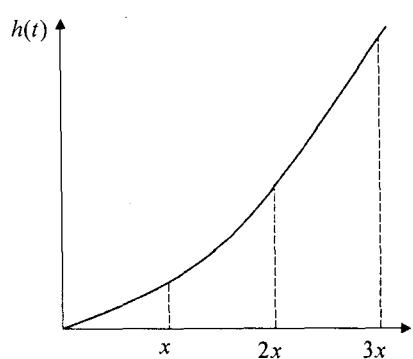
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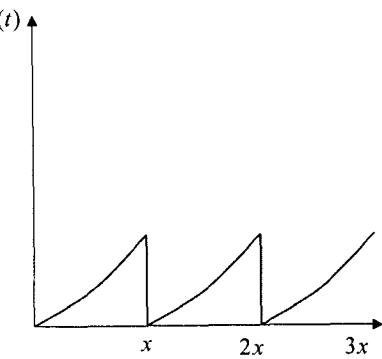
I. Introduction

- Types of repair

Minimal repair



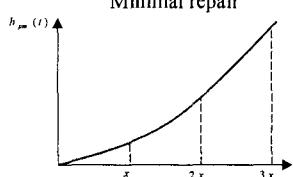
Perfect repair



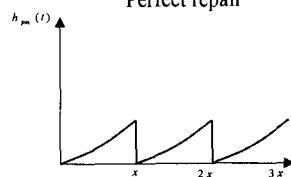
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Minimal repair

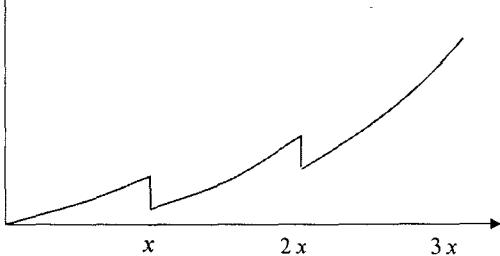


Perfect repair



$h(t)$

Imperfect repair



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- Imperfect repair
 - Chan and Downs(1978)
 - Nakagawa(1979)
 - Murthy and Nguyen(1981)
 - Canfield(1986)
 - Doyen and Gaudoin(2004)

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- Doyen and Gaudoin imperfect repair model

– ARI_1 imperfect repair model

$$h_{re}(t) = h(t) - ph(T_{N_t}), \quad 0 \leq p \leq 1,$$

– ARI_∞ imperfect repair model

$$h_{re}(t) = h(t) - p \sum_{j=0}^{N_t-1} (1-p)^j h(T_{N_t-j}), \quad 0 \leq p \leq 1,$$

where N_t is the number of failure observed up to time t ,
and $h(t)$ is initial failure rate.

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- Purpose of this paper

1. To develop the two periodic preventive maintenance models based on ARI₁ and ARI_∞ imperfect repair models.
2. To derive the expected cost rate per unit time for two PM models.
3. To obtain the optimal periodic PM schedules. That is, to obtain the optimal number of PM's and optimal period which minimize the expected cost rate per unit time
4. To make a comparison between the optimal PM schedule of the periodic PM model based on ARI₁ imperfect repair model and one of the periodic PM model based on ARI_∞ imperfect repair model

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- Notation

$h(t)$	hazard rate without PM
$h_{pm}(t)$	hazard rate with PM
x	period of PM
$N - 1$	number of PM's conducted before replacement
p	improvement factor in hazard rate, $0 \leq p \leq 1$
C_{mr}	cost of minimal repair at failure
C_{pm}	cost of PM
C_{re}	cost of replacement
$C(x, N)$	expected cost rate per unit time

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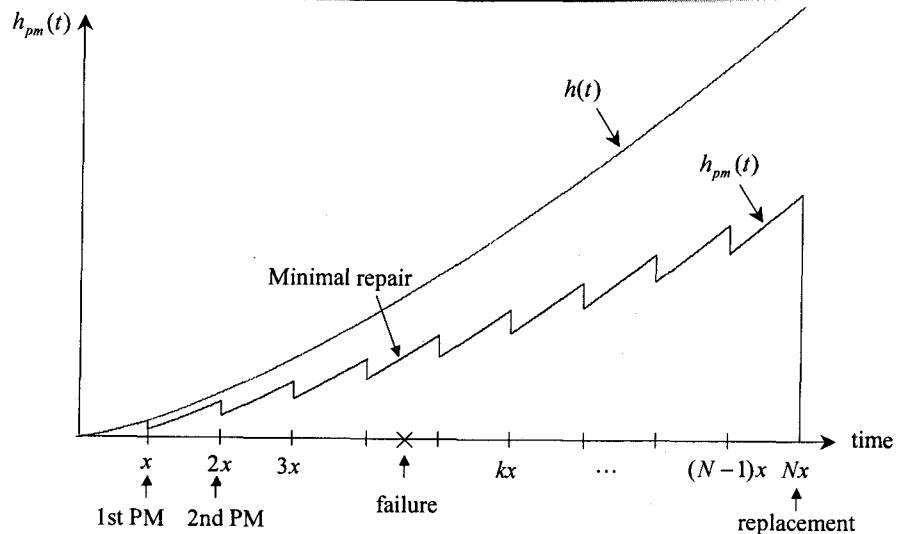
II. Model and assumption

- Assumptions

- The system begins to operate at time $t = 0$.
- The PM is done at periodic time kx ($k = 1, 2, \dots, N-1; x > 0$), and the system is replaced by new one at the N th PM.
- Each PM reduces the hazard rate by a certain amount. Under the periodic PM model based on ARI_1 (hereafter, PM model 1), each PM reduces the hazard rate increased since the last PM. Under the periodic PM model based on ARI_∞ (hereafter PM model 2), each PM reduces the hazard rate of an proportional to the current the hazard rate.
- The system undergoes only minimal repair at failures between PM's
- Times for repair and PM are negligible.
- $h(t)$ is strictly increasing and convex function in $t \geq 0$

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• Model description

1. Model 1

$$h_{pm}(t) = \begin{cases} h(t) & 0 < t \leq x \\ h(t) - ph(kx) & kx < t \leq (k+1)x \end{cases} \quad (1)$$

2. Model 2

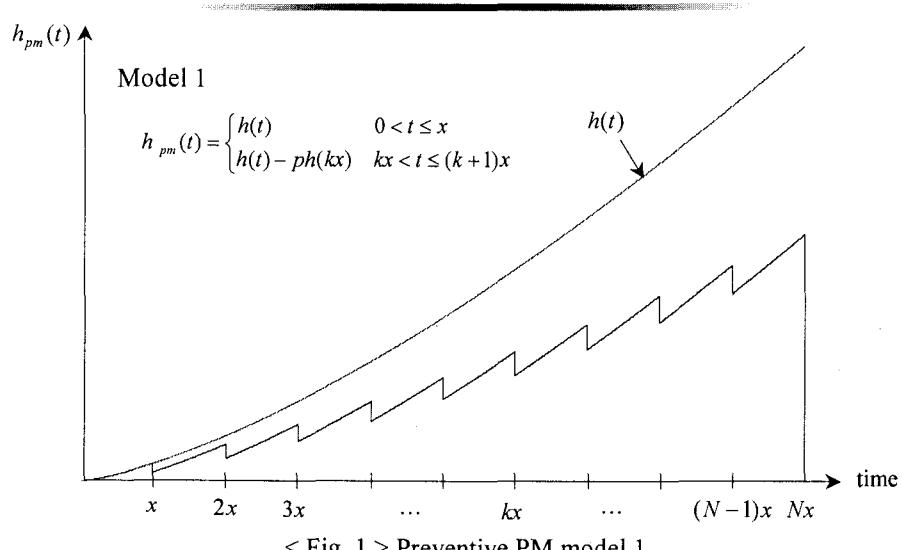
$$h_{pm}(t) = \begin{cases} h(t) & 0 < t \leq x \\ h(t) - p \sum_{j=0}^{k-1} (1-p)^j h((k-j)x) & kx < t \leq (k+1)x \end{cases} \quad (2)$$

for $k = 1, 2, \dots, N-1$, $0 \leq p \leq 1$, $h_{pm}(0) = h(0)$

When $p = 0$, the system after PM is as bad as old one while when $p = 1$, the system after PM is as good as new one.

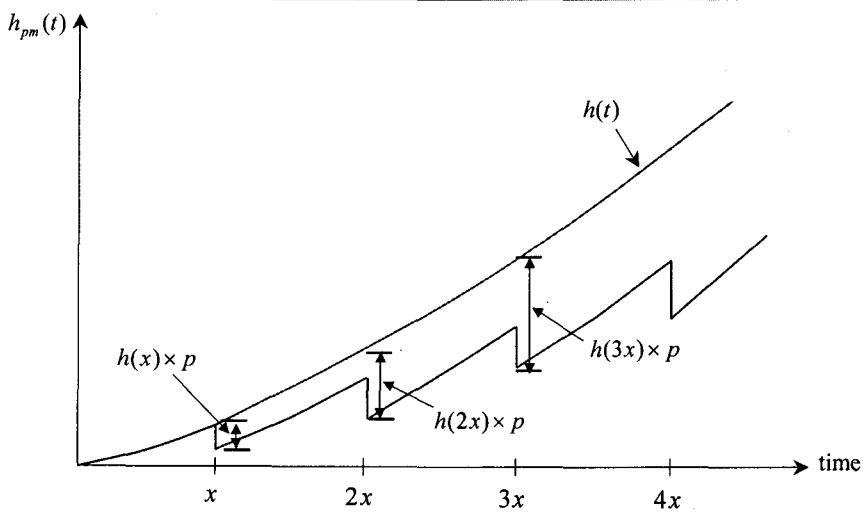
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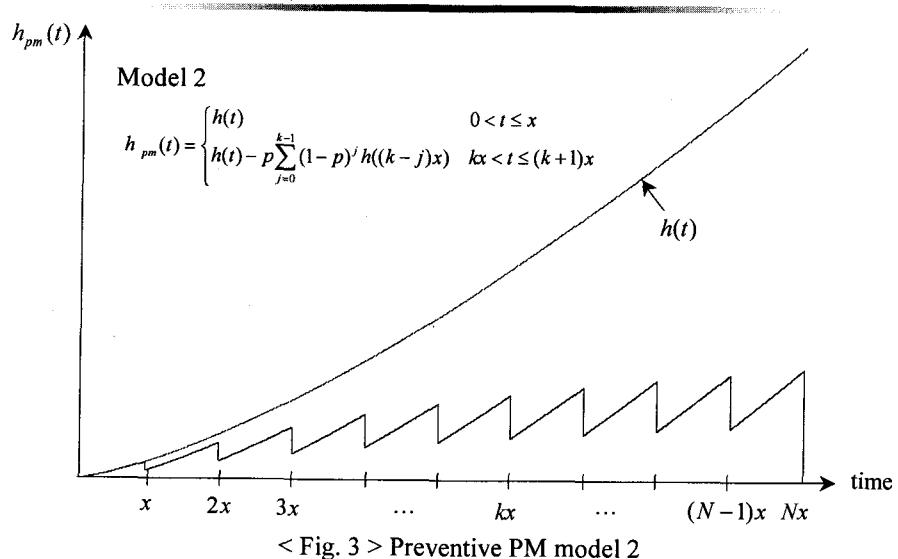
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< Fig. 2 > Preventive PM model 1

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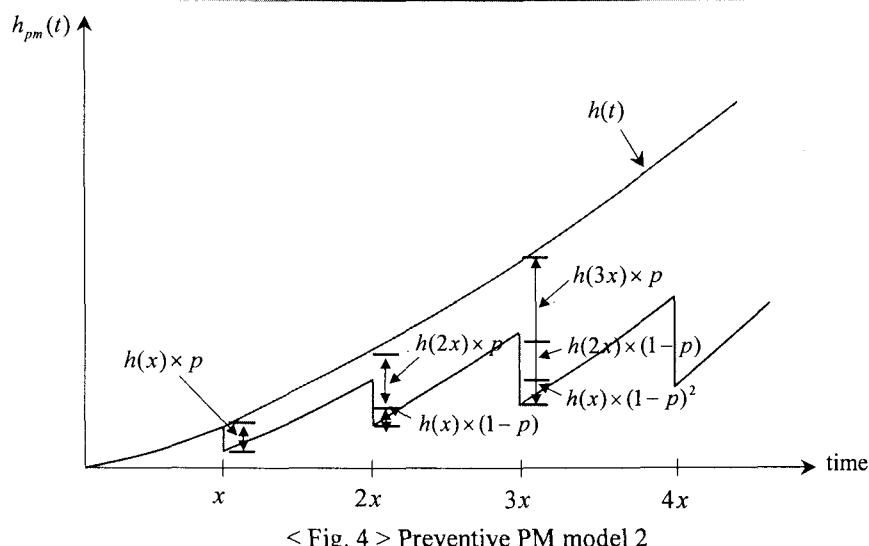
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< Fig. 3 > Preventive PM model 2

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< Fig. 4 > Preventive PM model 2

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III. Expected cost rate per unit time

- Expected cost of minimal repair in $[0, Nx]$

$$= C_{mr} \left(\sum_{k=0}^{N-1} \int_{kx}^{(k+1)x} h_{pm}(t) dt \right)$$

- Expected cost of preventive maintenance in $[0, Nx]$

$$= (N-1)C_{pm}$$

- Expected cost of replacement in $[0, Nx]$

$$= C_{re}$$

⇒ Expected cost rate per unit time

$$C(x, N) = \frac{1}{Nx} \left[C_{mr} \int_0^{Nx} h_{pm}(t) dt + (N-1)C_{pm} + C_{re} \right]. \quad (3)$$

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Substituting $h_{pm}(t)$ in the equation (3), by the equation (1) and (2), we obtain two expected cost rate as follows.

1. Expected cost rate per unit time in Model 1

$$C_1(x, N) = \frac{1}{Nx} \left[C_{mr} \left\{ H(Nx) - px \sum_{k=0}^{N-1} h(kx) \right\} + (N-1)C_{pm} + C_{re} \right]$$

2. Expected cost rate per unit time in Model 2

$$C_2(x, N) = \frac{1}{Nx} \left[C_{mr} \left\{ H(Nx) - px \sum_{k=0}^{N-1} \sum_{j=0}^{k-1} (1-p)^j h((k-j)x) \right\} + (N-1)C_{pm} + C_{re} \right]$$

where $H(t) = \int_0^t h(x) dx$

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IV. Optimal schedules for the periodic PM policies

4.1 Optimal number of PM(when PM period x is known in model 1)

To find the optimal number N^* , we form be following inequalities

$$C_1(x, N+1) \geq C_1(x, N) \quad \text{and} \quad C_1(x, N) < C_1(x, N-1) \quad (4)$$

For $0 \leq p \leq 1$, $C_1(x, N+1) \geq C_1(x, N)$ implies

$$L_1(x, N) \left[N \{H((N+1)x) - H(Nx)\} - H(Nx) - Npxh(Nx) + px \sum_{k=0}^{N-1} h(kx) \right] \geq \frac{C_{re} - C_{pm}}{C_{mr}}$$

$C_1(x, N) < C_1(x, N-1)$ implies

$$(N-1) \{H(Nx) - H((N-1)x)\} - H((N-1)x) - (N-1)pxh((N-1)x) + px \sum_{k=0}^{N-2} h(kx) < \frac{C_{re} - C_{pm}}{C_{mr}}$$

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4.1 Optimal number of PM(when PM period x is known in model 1)

- Lemma 1

If $h(t)$ is convex and strictly increasing in $t \geq 0$,

then $L_1(x, N)$ is increasing function in N

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proof > Let $m(k) = \int_{kx}^{(k+1)x} h(t)dt - pxh(kx) = H((k+1)x) - H(kx) - pxh(x)$

and $\xi(k) = \int_{kx}^{(k+1)x} h(t)dt - xh(kx)$.

Then, for all $k \in N$, we have

$$\frac{x^2 h'(kx)}{2} \leq \int_{kx}^{(k+1)x} h(t)dt - xh(kx) \leq \frac{x\{h((k+1)x) - h(kx)\}}{2}. \quad (5)$$

It is note that the left - hand side and the right - hand side inequality of (5), respectively, is increasing in k since $h(t)$ is convex.

Therefore, $\xi(k) = \int_{kx}^{(k+1)x} h(t)dt - xh(kx)$ is increasing in k . Then

$$m(k) = \xi(k) + (1-p)xh(kx).$$

Since both $\xi(k)$ and $h(kx)$ is increasing in k , $m(k)$ is also increasing in k for $0 \leq p \leq 1$.

Thus, we have

$$\begin{aligned} L_1(x, N) - L_1(x, N-1) \\ &= N[\{H((N+1)x) - H(Nx)\} - \{H(Nx) - H((N-1)x)\}] \\ &\quad - \{pxh(Nx) - pxh((N-1)x)\} \\ &= N[m(N) - m(N-1)] > 0 \end{aligned}$$

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- Theorem 2

If $h(t)$ is convex and strictly increasing in $t \geq 0$, then there exists a finite and unique N^* which satisfies (4) for any $x > 0$.

proof > It is shown from Lemma 1, that $L_1(x, N)$ is increasing in N .

From equation (5), we have

$$\begin{aligned} L_1(x, N) &= \sum_{k=0}^{N-1} \left[\left\{ \int_{Nx}^{(N+1)x} h(t) dt - pxh(Nx) \right\} - \left\{ \int_x^{(k+1)x} h(t) dt - pxh(kx) \right\} \right] \\ &\geq \sum_{k=0}^{N-1} \left[\frac{x^2 h'(Nx)}{2} - \frac{x \{h((k+1)x) - h(kx)\}}{2} \right] \\ &= \frac{x}{2} [Nxh'(Nx) - h(Nx)] \end{aligned}$$

which becomes ∞ as $N \rightarrow \infty$.

Thus there exists a finite and unique N^* which satisfies (4) for any $x > 0$.

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4.2 Optimal period of PM (when PM number N is known in model 1)

Differentiation $C_1(x, N)$ with respect to x and set it to zero, then we have

$$\frac{Nxh(Nx) - H(Nx) - px^2 \sum_{k=0}^{N-1} kh'(kx)}{g_1(x)} = \frac{(N-1)C_{pm} + C_{re}}{C_{mr}} \quad (6)$$

$$\frac{d}{dx} C_1(x, N) = g_1(x) - C$$

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4.2 Optimal period of PM(when PM number N is known in model 1)

- Lemma 3

Suppose that $N^2 x h'(Nx) > px \sum_{k=0}^{N-1} (2kh'(kx) + k^2 x h''(kx))$

Then $g_1(x)$ is increasing in $x > 0$.

proof > Taking derivative of $g_1(x)$, with respect to x yields

$$\frac{dg_1(x)}{dx} = N^2 x h'(Nx) - 2px \sum_{k=0}^{N-1} kh'(kx) - px^2 \sum_{k=0}^{N-1} k^2 h''(kx)$$

Then, it is clear that $\frac{dg_1(x)}{dx} > 0$, and $g_1(x)$ is increasing in x .

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- Theorem 4

Suppose that there exists a $x_0 > 0$, such that

$$g_1(x) > \frac{(N-1)C_{pm} + C_{re}}{C_{mr}}$$

for all $x > x_0$. Then the optimal period x^* exist and unique.

proof > Note that $g_1(x) = 0$ at $x = 0$. Since from Lemma 3, $g_1(x)$ is increasing and $g_1(x) > \frac{(N-1)C_{pm} + C_{re}}{C_{mr}}$, there exists x^* which satisfies equation (6) and it is unique.

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4.3 Optimal number of PM(when PM period x is known in model 2)

To find the optimal number N^* , we form the following inequalities

$$C_2(x, N+1) \geq C_2(x, N) \quad \text{and} \quad C_2(x, N) < C_2(x, N-1) \quad (7)$$

For $0 \leq p \leq 1$, $C_2(x, N+1) \geq C_2(x, N)$ implies

$$\begin{aligned} L_2(x, N) & \left[N\{H((N+1)x) - H(Nx)\} - H(Nx) - Npx \sum_{j=0}^{N-1} (1-p)^j h((N-j)x) \right. \\ & \quad \left. + px \sum_{k=0}^{N-1} \sum_{j=0}^{k-1} (1-p)^j h((k-j)x) \right] \geq \frac{C_{re} - C_{pm}}{C_{mr}} \end{aligned}$$

$C_2(x, N) < C_2(x, N-1)$ implies

$$\begin{aligned} (N-1)\{H(Nx) - H((N-1)x)\} - H((N-1)x) - (N-1)px \sum_{j=0}^{N-2} (1-p)^j h((N-j-1)x) \\ + px \sum_{k=0}^{N-2} \sum_{j=0}^{k-1} (1-p)^j h((k-j)x) < \frac{C_{re} - C_{pm}}{C_{mr}}. \end{aligned}$$

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4.3 Optimal number of PM(when PM period x is known in model 2)

- Lemma 5

Let $\xi(k) = \int_{kx}^{(k+1)x} h(t)dt - x \sum_{j=0}^{k-1} (1-p)^j h((k-j)x)$. Suppose that

$\xi(k)$ is increasing in k . If $h(t)$ is strictly increasing in $t \geq 0$, then

$L_2(x, N)$ is increasing in N .

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proof > For $t \geq 0$, it is easy to see that

$$\begin{aligned} L_2(x, N) - L_2(x, N-1) \\ = N[\{H((N+1)x) - H(Nx)\} - \{H(Nx) - H((N-1)x)\} \\ - \left\{ px \sum_{j=0}^{N-1} (1-p)^j h((N-j)x) - px \sum_{j=0}^{N-2} (1-p)^j h((N-j-1)x) \right\}]. \end{aligned}$$

$$\begin{aligned} \text{Let } m(k) = \int_{kx}^{(k+1)x} h(t) dt - px \sum_{j=0}^{k-1} (1-p)^j h((k-j)x) \\ = H((k+1)x) - H(kx) - px \sum_{j=0}^{k-1} (1-p)^j h((k-j)x) \end{aligned}$$

$$\text{Then, } m(k) = \xi(k) + (1-p)x \sum_{j=0}^{k-1} (1-p)^j h((k-j)x).$$

Since both $\xi(k)$ and $h(kx)$ is increasing in k , $m(k)$ is also increasing in k for $0 \leq p \leq 1$.

Thus, we have

$$L_2(x, N) - L_2(x, N-1) = N[m(N) - m(N-1)] > 0.$$

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- Theorem 6

Let $\xi(k) = \int_{kx}^{(k+1)x} h(t) dt - x \sum_{j=0}^{k-1} (1-p)^j h((k-j)x)$. Suppose that

$\sum_{k=0}^{N-1} [\xi(N) - \xi(k)] \geq \frac{C_{re} - C_{pm}}{C_{mr}}$ as $N \rightarrow \infty$. Then there exists

a finite and unique N^* satisfies the equation (7).

proof > Since $0 \leq p \leq 1$, we have $L_2(x, N) \geq \sum_{k=0}^{N-1} [\xi(N) - \xi(k)]$ for all N .

Note that $L_2(x, N) = 0$ when $N = 0$ and $L_2(x, N)$ is increasing in N .

Since

$$\sum_{k=0}^{N-1} [\xi(N) - \xi(k)] \geq \frac{C_{re} - C_{pm}}{C_{mr}} \text{ as } N \rightarrow \infty$$

$$L_2(x, N) \geq \frac{C_{re} - C_{pm}}{C_{mr}} \text{ as } N \rightarrow \infty,$$

which completes the proof.

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4.4 Optimal period of PM(when PM number N is known in model 2)

Differentiation $C_2(x, N)$ with respect to x and set it to zero,
then we have

$$\frac{Nxh(Nx) - H(Nx) - px^2 \sum_{k=0}^{N-1} \sum_{j=0}^{k-1} (1-p)^j (k-j) h'((k-j)x)}{g_2(x)} = \frac{(N-1)C_{pm} + C_{re}}{C}$$

(8)

$$\frac{d}{dx} C_2(x, N) = g_1(x) - C$$

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4.4 Optimal period of PM(when PM number N is known in model 2)

- Lemma 7**

Suppose that $N^2 x h'(Nx) > 2px \sum_{k=0}^{N-1} \sum_{j=0}^{k-1} 2(1-p)^j (k-j) h'((k-j)x)$
 $+ px^2 \sum_{k=0}^{N-1} \sum_{j=0}^{k-1} (1-p)^j (k-j)^2 h''((k-j)x)$.

Then $g_2(x)$ is increasing in $x > 0$.

proof > Taking derivative of $g_2(x)$ with respect to x yields

$$\begin{aligned} \frac{dg_2(x)}{dx} &= N^2 x h'(Nx) - 2px \sum_{k=0}^{N-1} \sum_{j=0}^{k-1} (1-p)^j (k-j) h'((k-j)x) \\ &\quad - px^2 \sum_{k=0}^{N-1} \sum_{j=0}^{k-1} (1-p)^j (k-j)^2 h''((k-j)x) \end{aligned}$$

Then, it is clear that $\frac{dg_2(x)}{dx} > 0$ and $g_2(x)$ is increasing in x .

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- Theorem 8

Suppose that there exists a $x_0 > 0$, such that

$$g_2(x) > \frac{(N-1)C_{pm} + C_{re}}{C_{mr}}$$

for all $x > x_0$. Then the optimal period x^* exist and unique.

proof > Note that $g_2(x) = 0$ at $x = 0$. Since from Lemma 7,

$g_2(x)$ is increasing and $g_2(x) > \frac{(N-1)C_{pm} - C_{re}}{C_{mr}}$, there exists x^*

which satisfies equation (8) and it is unique.

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V . Numerical examples

- Optimal number of PM's N^* and its expected cost rate $C(x, N^*)$

P	C_{re}						$C_1(x, N^*)$	
	2.0		2.5		3.0			
	N^*	$C(x, N^*)$	N^*	$C(x, N^*)$	N^*	$C(x, N^*)$		
0.1	1	3.2651	1	3.8901	2	4.4860	2	4.7985
0.2	1	3.2651	1	3.8901	2	4.4019	2	4.7144
0.3	1	3.2651	1	3.8901	2	4.3177	2	4.6302
0.4	1	3.2651	1	3.8901	2	4.2336	2	4.5461
0.5	1	3.2651	2	3.8369	2	4.1494	2	4.4619
0.6	1	3.2651	2	3.7527	2	4.0652	2	4.3777
0.7	1	3.2651	2	3.6686	2	3.9811	3	4.2726
0.8	1	3.2651	2	3.5844	3	3.8972	3	4.0876
0.9	2	3.1878	3	3.4859	3	3.6942	4	3.8740
1.0	3	3.0926	5	3.2582	7	3.3625	9	3.4400
$C_2(x, N^*)$								
0.1	1	3.2651	1	3.8901	2	4.4860	2	4.7985
0.2	1	3.2651	1	3.8901	2	4.4019	2	4.7144
0.3	1	3.2651	1	3.8901	2	4.3177	2	4.6302
0.4	1	3.2651	1	3.8901	2	4.2336	2	4.5461
0.5	1	3.2651	2	3.8369	2	4.1494	2	4.4619
0.6	1	3.2651	2	3.7527	2	4.0652	3	4.3229
0.7	1	3.2651	2	3.6686	3	3.9464	4	4.1482
0.8	1	3.2651	3	3.5811	4	3.7783	5	3.9206
0.9	2	3.1878	4	3.4328	5	3.5742	7	3.6762
1.0	3	3.0926	5	3.2582	7	3.3625	9	3.4400

$$h(t) = \beta \lambda^{\beta-1} t^{\beta-1}$$

$$\beta > 0, t > 0$$

$$\beta = 2.2, \lambda = 1$$

$$C_{mr} = 1, C_{pm} = 1.5$$

$$x = 0.8$$

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- Optimal period of PM's x^* and its expected cost rate $C(x^*, N)$

N	P							
	0.1		0.3		0.6		1.0	
	x^*	$C(x^*, N)$						
$C_1(x^*, N)$								
1	1.5166	3.6264	1.5166	3.6264	1.5166	3.6264	1.5166	3.6264
3	0.7142	5.1342	0.7642	4.7982	0.8662	4.2333	1.1122	3.2969
5	0.5187	6.3618	0.5647	5.8438	0.6677	4.9426	1.0058	3.2810
7	0.4235	7.4205	0.4647	6.7625	0.5613	5.5989	0.9514	3.3035
9	0.3652	8.3668	0.4026	7.5904	0.4925	6.2036	0.9165	3.3338
11	0.3249	9.2317	0.3593	8.3503	0.4434	6.7659	0.8915	3.3651
13	0.2951	10.034	0.3269	9.0572	0.4061	7.2932	0.8723	3.3952
15	0.2719	10.786	0.3017	9.7212	0.3764	7.7914	0.8567	3.4237
17	0.2533	11.497	0.2813	10.349	0.3523	8.2649	0.8438	3.4506
19	0.2378	12.173	0.2644	10.948	0.3321	8.7172	0.8328	3.4757
$C_2(x^*, N)$								
1	1.5166	3.6264	1.5166	3.6264	1.5166	3.6264	1.5166	3.6264
3	0.7204	5.0899	0.7826	4.6851	0.8983	4.0819	1.1122	3.2969
5	0.5354	6.1634	0.6154	5.3625	0.7593	4.3461	1.0058	3.2810
7	0.4481	7.0132	0.5394	5.8264	0.6965	4.5121	0.9514	3.3035
9	0.3961	7.7139	0.4956	6.1641	0.6597	4.6314	0.9165	3.3338
11	0.3611	8.8165	0.4671	6.4219	0.6349	4.7248	0.8915	3.3651
13	0.3359	9.8165	0.4469	6.6265	0.6167	4.8021	0.8722	3.3952
15	0.3167	9.2611	0.4317	6.7938	0.6025	4.8681	0.8568	3.4237
17	0.3016	9.6528	0.4198	6.9344	0.5911	4.9259	0.8438	3.4506
19	0.2894	10.000	0.4103	7.0550	0.5816	4.9776	0.8328	3.4757

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- Comparison between model 1 and model 2

- Optimal number of PM is smaller when $C_1(x, N^*)$ than $C_2(x, N^*)$ for given x .
- Because PM effect of model 2 is greater than PM effect of model 1
- Optimal period of PM is smaller when $C_1(x^*, N)$ than $C_2(x^*, N)$ for given N .
- Because PM effect of model 2 is greater than PM effect of model 1

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