ON ANALYTICAL SOLUTION OF NON LINEAR ROLL EQUATION OF SHIPS

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Abstract

Out of all types of motions the critical motions leading to capsize is roll. The dynamic amplification in case of roll motion may be large for ships as roll natural frequency generally falls within the frequency range of wave energy spectrum typical used for estimation of motion spectrum. Roll motion is highly non-linear in nature. There are various representations of non-linear damping and restoring available in literature. In this paper an uncoupled non-linear roll equations with three representation of damping and cubic restoring term is solved using a perturbation technique. Damping moment representations are linear plus quadratic velocity damping, angle dependant damping and linear plus cubic velocity dependant damping. Numerical value of linear damping coefficient is almost same for all types but non-linear damping is different. Linear and non-linear damping coefficients are obtained form free roll decay tests. External rolling moment is assumed as deterministic with sinusoidal form. Maximum roll amplitude of non-linear roll equation with various representations of damping is calculated using analytical procedure and compared with experimental results, which are obtained from forced tests in regular waves by varying frequency with three wave heights. Experiments indicate influence of non-linearity at resonance frequency. Both experiment and analytical results indicates increase in maximum roll amplitude with wave slope at resonance. Analytical results are compared with experiment results which indicate maximum roll amplitude analytically obtained with angle dependent and cubic velocity damping are equal and difference from experiments with these damping are less compared to non-linear equation with quadratic velocity damping.

Keywords: maximum roll amplitude; non-linear damping; non-linear restoring; innovative classical perturbation technique

Nomenclature

I	Transverse moment of inertia
A_{44}	Added moment of inertia
$B(\phi,\dot{\phi})$	Damping moment
$B_{iL(i=1,2,3)}$	Linear damping coefficient
$B_{iNL(i=1,2,39}$	Nonlinear damping coefficient

$M(\phi,t)$	Restoring moment of ship
K_1	Linear restoring moment
K_2	Nonlinear restoring
$E(\phi,t)$	External force moment
ω_ϕ	Roll natural frequency
ω_e	Wave encounter frequency
α_m	Wave slope
λ	Nondimensionalised moment of inertia
γ	Effective wave slope coefficient
Δ	Displacement

1. Introduction

Out of six motions of ship, rolling has been in main attention over the years because of it being most critical motion leading to capsize. For most degrees of freedom, radiation damping is enough for estimation of ship motions at wave frequencies. It is well know that roll damping is highly non-linear for ships. Nonlinearities are due to nature of damping and restoring moment. In case of small angles response can be well estimated by a linear equation. When amplitude increases nonlinearites becomes important. Many authors suggested various representations for non-linear damping and restoring moment. Restoring moments is usually represented by various degree of odd order polynomial. Non-linear damping term representations vary much. Classical damping theory takes into account quadratic dependency from the angular velocity. This representation is usually known as conventional linear plus quadratic velocity damping. Dalzal (1978) suggested various representation of damping moment. He introduced a cubic dependency. This is referred as linear plus cubic velocity damping. Antonio Cardo *et al.* (1982) gave coherent damping models taking into dependency of roll angle on damping. Haddara (1989) studied angle dependence of roll damping. Various linear and nonlinear formulations of roll motions and devices for reducing roll motions were discussed by Bhattacharyya (1978).

Researchers favored different representations for damping moment in their studies for either numerical or analytical solutions of nonlinear equation. Taylon (2000) solved non-linear uncoupled roll equation in frequency domain using Duffing method. He used three widely used damping models for obtaining analytical solution. Surendran and Reddy (2003) solved linear and non-linear roll uncoupled equation to obtain roll response in regular waves using various types of damping and restoring forces. Surendran *et al.* (2005) analytically solved non-linear equation and showed multi-peak phenomena in roll response curves. Taylon (1999) used generalized asymptotic method to solve non-liner roll equation in time domain with linear plus quadratic velocity depend damping and cubic restoring force.

In spite of these developments it is still difficult to decide on, in design stage, what type of non-linear damping and restoring model should take in to consideration, solution procedure out of variety of techniques. Linear and non-linear damping coefficients can be estimated from analysis of free roll decay curve. It is know that different damping models/representations may be obtained from the same roll decay data Haddara (1989). Thus a criterion for selection of model will be how well it predicts roll response. Obtained roll responses may also depend on technique used for obtaining solution. Thus a correct non-linear representation or solution technique may be one, which accurately estimates maximum roll amplitude when compared with experiment results.

The main object of this paper is aimed in this direction. Various nonlinear differential equations representing roll motion of ship available in literature are solved analytically using innovative classical perturbation technique. Experiments are carried out at zero speed using a ship model. Model was forced to roll in regular waves. Experiments are carried out with varying wave slope to study its influence on response. Maximum roll amplitude obtained from analytical and experiments are compared. Damping and non-linear damping coefficients in each type of roll equation are calculated from analysis of free roll data.

2. Mathematical model

When rolling motion is concerned to be independent, the following equation describing decoupled roll motion may be represented as follows:

$$(I + A_{44})\ddot{\phi} + B(\phi,\dot{\phi}) + M(\phi,t) = E(\phi,t).$$

This equation may be written to various forms. In the present study, three different types of nonlinear damping are considered. Cubic expression is taken for restoring term in all equations. There are two expressions generally used for right side forcing term. One expression depends on square of natural roll frequency ϖ_{φ} (Giorgio Contento,1996) while other takes square of wave encounter frequency ϖ_e (Taylon,2000). In this study forcing term based on natural roll frequency is considered. Following is list of equations after dividing throughout with total moment of inertia term. These equations are named as B1, B2 and B3 types after Taylon (2002)

B1 type non-linear roll equation (linear plus quadratic velocity damping):

$$(I_{44} + A_{44})\ddot{\phi} + B_{1L}\dot{\phi} + B_{1NL}\dot{\phi}\left|\dot{\phi}\right| + \Delta(K_1\phi + K_3\phi^3) = \gamma\omega_{\phi}^2\alpha_m I_{44}\cos\omega_e t$$

B2 type (angle dependant damping):

$$(I_{44} + A_{44})\ddot{\phi} + B_{2L}\dot{\phi} + B_{2NL}\phi^2\dot{\phi} + \Delta(K_1\phi + K_3\phi^3) = \gamma\omega_{\phi}^2\alpha_m I_{44}\cos\omega_e t$$

B3 type (linear plus cubic velocity dependant damping):

$$(I_{44} + A_{44})\ddot{\phi} + B_{3L}\dot{\phi} + B_{3NL}\dot{\phi}^3 + \Delta(K_1\phi + K_3\phi^3) = \gamma\omega_{\phi}^2\alpha_m I_{44}\cos\omega_e t$$

Where overdots denote the differentiation with respect to time 't'. Constants 'K_1' and 'K_3' can be obtained by righting arm curve at a particular loading condition. Here 'K_1' equals to metacenter GM of ship.

3. Experiment programme:

Experiments have been conducted at the ship maneuvering basin of Tokyo University of Marine Science and Technology, Tokyo, Japan. For the experiments, a wooden ship model of the training ship SHIOJI MARU has been used. The model is fitted with bilge keel. The model to ship scale ratio is 1:17.037. Principal dimensions of model and ship are shown in Table 1.

Roll decay tests at small roll amplitudes and forced roll tests in regular beam waves have conducted on scale model. Experiments are conducted with and without restrainment of heave motion to see influence of heave on roll if any.

Forced roll in regular beam waves were conducted by varying both wave frequency and wave height. Tests have been conducted by varying wave frequency at constant wave height. The wave frequency has been varied from 0.4 to 1.5 Hz during each case of constant wave height. Experiments are conducted with wave heights of 30mm, 40mm and 50mm at model scale. These wave heights correspond to 0.511m, 0.681m and 0.851m at full scale. Ratio of wave height to draft is 0.17, 0.22 and 0.28 respectively.

Table 1

Principal particulars of training ship SHIOJI MARU

Particulars	**Model**	**Ship**
Length between p.p (m)	2.7	45.0
Breadth, molded (m)	0.587	10.0
Depth, molded (m)	0.358	6.1
Draft molded (m)	0.176	3.0
C.G above baseline (m)	0.1946	3.31
Displacement (ton)	0.150	785
Trans. radius of Gyration (k_{xx})	0.2054	3.5

4.Estimation of damping coefficients

Decrement and extinction curves obtained from free roll decay analysis were used to estimate the damping coefficients in assumed roll equations given in Sec 2. Numerical values of linear and nonlinear damping coefficients depend upon form of damping moment presented in non-linear equation. This may be a better approach than assuming same values for all equations. Use of constant values for linear and nonlinear damping coefficients, irrespective of form of damping representation, is one of procedures found in literature. In this present study, it is assumed that the numerical values of damping coefficients vary with damping representation in roll equation. Extinction curve may be fitted to linear, quadratic or cubic form and thus linear and nonlinear damping coefficients may be calculated separately. Table 2 shows numerical values of damping and nonlinear damping coefficients.

Table 2

Numerical values of linear and nonlinear damping coefficients.

Type of roll equation	Linear damping	Non-linear damping
B1type non-linear	0.0822	0.0225
B2 type	0.08375	0.1383
B3 type	0.08375	0.0465

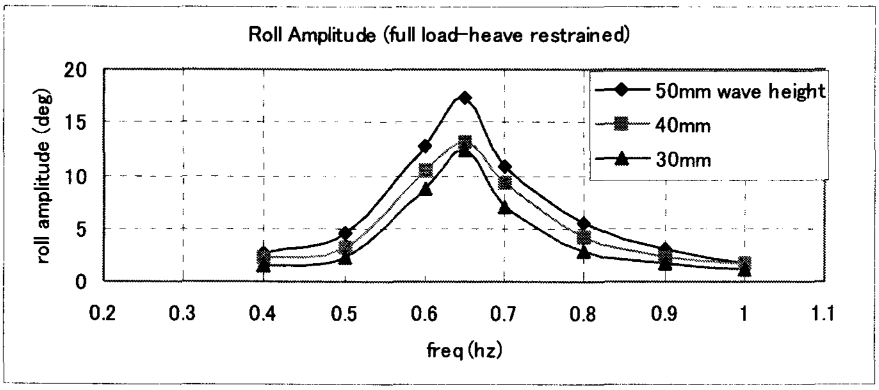

Fig.1. Roll response curve from experiments

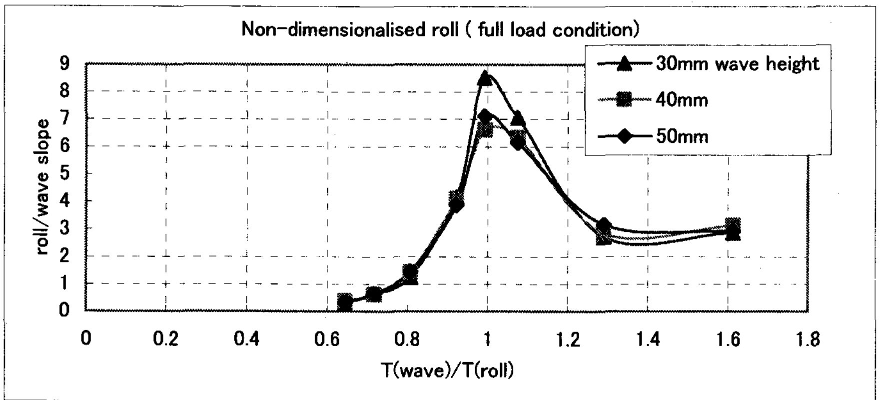

Fig.2 Non-dimensionalised roll response (ratio of wave height to draft is 0.17,0.22 and 0.28 respectively for 30mm,40mm, 50mm wave height)

5. Analytical solution of roll equation

Innovative classical perturbation technique was used to solve non-linear roll equations given in Sec.2. It is possible to obtain solution in time domain. In this method the fundamental frequency is represented differently from the classical perturbation technique. However the solution has been terminated after obtaining a sixth order polynomial of roll amplitude ϕ_a.

A non-linear differential equation can be considered of the form

$$\ddot{\phi} + \phi = \mu f(t, \phi, \dot{\phi}) \tag{4}$$

Where μ is non-dimensional parameter associated with the nonlinear terms, assumed to be small.

The solution technique using innovative classical perturbation technique is given for B1 type nonlinear differential equation. A second order B1 type nonlinear differential equation as given in Sec.2 is:

$$(I_{44} + A_{44})\ddot{\phi} + B_{1L}\dot{\phi} + B_{1NL}\dot{\phi}\left|\dot{\phi}\right| + \Delta(K_1\phi + K_3\phi^3) = \gamma\omega_\phi^2\alpha_m I_{44}\cos\omega_e t \tag{5}$$

If Eqs.(5) is divided throughout by $(I_{44} + A_{44})$ to take following form:

$$\ddot{\phi} + b_{1L}\dot{\phi} + b_{1NL}\dot{\phi}\left|\dot{\phi}\right| + \omega_\phi^2\phi + k_3\phi^3 = \gamma\lambda\omega_\phi^2\alpha_m\cos\omega_e t \tag{6}$$

Where

$$\omega_\phi^2 = \frac{\Delta GM}{(I_{44}+A_{44})};\ b_{1L} = \frac{B_{1L}}{(I_{44}+A_{44})};\ b_{1NL} = \frac{B_{1NL}}{(I_{44}+A_{44})};\ k_3 = \frac{\Delta K_3}{(I_{44}+A_{44})};\ \lambda = \frac{I}{(I_{44}+A_{44})} \tag{7}$$

γ = Effective wave slope

Eqs.(5) can be considered as a special case of Duffing equation [Taylon,2000]. Roll displacement and the impressed force can be expected to be out of phase due to presence of damping, as in the case of the corresponding simple linear problem. It is more convenient to fix the phase of the solution and leave the phase of the impressed force as a quantity to be determined. Hence, Eqs.(6) can be expressed as:

$$\ddot{\phi} + b_L\dot{\phi} + b_{NL}\dot{\phi}|\dot{\phi}| + \omega_\phi^2\phi + k_3\phi^3 = H\cos\omega_e t - G\sin\omega_e t \tag{8}$$

A periodic solution $\phi(t)$ is desired which has same frequency as external moment $H\cos\omega_e t - G\sin\omega_e t$. To avoid working with functions of unknown period, a new independent variable 'τ' is used to replace 't' through the relation $\tau = \omega_e t$. Hence the Eqs.(8) becomes

$$\omega_e^2\ddot{\phi} + b_L\omega_e\dot{\phi} + b_{NL}\omega_e^2\dot{\phi}|\dot{\phi}| + \omega_\phi^2\phi + k_3\phi^3 = H\cos\tau - G\sin\tau \tag{9}$$

Where single dot represents differentiation with respect to new variable 'τ' Following the procedure of perturbation techniques above equation is written in the following form using the small parameter 'μ' associated with nonlinear terms. The external force is also considered of order of small parameter μ.

$$\omega_e^2\ddot{\phi} + \omega_\phi^2\phi + \mu(\omega_e b_L\dot{\phi} + b_{NL}\omega_e^2\dot{\phi}|\dot{\phi}| + k_3\phi^3) = \mu(H\cos\tau - G\sin\tau) \tag{10}$$

The solution of above equation is assumed in the form

$$\phi(\tau) = \phi_0(\tau) + \mu\phi_1(t) + \mu^2\phi_2(t) + \ldots\ldots.. \tag{11}$$

The fundamental frequency ω_e^2 is given by (Innovative classical perturbation technique, H.Hu, 2004)

$$w_e^2 = \omega_\phi^2 + \mu\omega_1 + \mu^2\omega_2 + \ldots\ldots\ldots \tag{12}$$

Substituting Eqs.(12) into Eqs.(10) and collection of terms with like powers of 'μ' gives the set of differential equations

$$\ddot{\phi}_0 + \phi_0 = 0$$
$$\omega_e^2\ddot{\phi}_1 + \omega_e^2\phi_1 = \omega_1\phi_0 - b_{1L}\omega_e\dot{\phi}_0 - b_{1NL}\omega_e^2\dot{\phi}_0|\dot{\phi}_0| - k_3\phi_0^3 \tag{13(1)-13(3)}$$
$$\omega_e^2\ddot{\phi}_2 + \omega_e^2\phi_2 = \omega_1\phi_1 - \omega_2\phi_0 - b_{1L}\omega_e\dot{\phi}_1 - b_{1NL}\omega_e^2\dot{\phi}_0|\dot{\phi}_1| - b_{1NL}\omega_e^2\dot{\phi}_0|\dot{\phi}_1|$$

Solving equation 13(1) by taking into account the initial conditions $\phi(0) = \phi_a$ and $\dot{\phi}(0) = 0$ gives

$$\phi_0 = \phi_a\cos\omega_e t$$

Substituting first order solution 'ϕ_0' on the right hand side of Eqs.13 (2), it becomes

$$\omega_e^2\ddot{\phi}_1 + \omega_e^2\phi_1 = \omega_1\phi_a\cos\omega_e t + b_L\omega_e\phi_a\sin\omega_e t + b_{NL}\omega_e^2\phi_a^2\sin\omega_e t|\sin\omega_e t| - k_3\phi_a^3\cos\omega_e t^3$$

Non- linear damping and 'k_3' term can be simplified as follows

$$-b_{NL}\dot{\phi}|\dot{\phi}| = b_{NL}\phi_a{}^2\omega^2{}_e \sin\omega_e t|\sin\omega_e t| = b_{NL}\phi_a{}^2\omega_e^2(8/3\pi)[\sin\omega_e t - (1/5)\sin 3\omega_e t - (1/35)\sin 5\omega_e t]$$

$$-k_3\phi^3 = -k_3(\phi_a{}^3\cos^3 wt) = -k_3\phi_a{}^3[(3/4)\cos\omega_e t + (1/4)\cos 3\omega_e t]$$

After neglecting the high harmonic term $\sin 5\omega_e t$, the differential equation for next approximate solution $\phi_1(t)$ may be written as follows:

$$\omega_e^2\ddot{\phi}_1 + \omega_e^2\phi_1 = [\phi_a\omega_1 - k_3(3/4)\phi_a{}^3 + H]\cos\omega_e t + [b_L\phi_a\omega_e + b_{NL}\phi_a{}^2\omega_e^2(8/3\pi) - G]\sin\omega_e t$$

$$-[b_{NL}\phi_a{}^2\omega_e^2(8/3\pi)(1/5)]\sin 3\omega_e t - [k_3\phi_a{}^3(1/4)]\cos 2\omega_e t$$

The right hand side contains terms in the form of $\sin\omega_e t$ and $\cos\omega_e t$. If these terms are presented the solution for the second order approximation would contain terms of the form $t\sin\omega_e t$ and $t\cos\omega_e t$. The periodicity condition for 'ϕ_1' requires that no secular terms should appear in the solution ϕ_1. Hence the following condition may be applied

$$[\phi_a\omega_1 - k_3(3/4)\phi_a{}^3] + [b_L\phi_a\omega_e + b_{NL}\phi_a{}^2\omega_e^2(8/3\pi)] = G\sin\omega_e t - H\cos\omega_e t$$

The right hand side of the equation may be written as $\lambda\alpha_m\omega_\phi^2$ [Taylon, 2000]. If effective wave slope 'γ' is also taken into account, the above condition may be written as

$$[\phi_a\omega_1 - k_3(3/4)\phi_a{}^3]^2 + [b_L\phi_a\omega_e + b_{NL}\phi_a{}^2\omega^2(8/3\pi)]^2 = \gamma\lambda^2\alpha_m^2\varpi_\phi^4$$

Inserting the expression for 'ω_1' in the above equation and considering only up to first order term of 'μ' and taking non-linearities in individual terms- b_{1L}, b_{1NL} and k_3, the above equation may be expanded to the following polynomial of roll amplitude ϕ_a.

$$[k_3^2(9/16)]\phi_a{}^6 + [(3/2)(\omega_\phi^2 - \omega_e^2)k_3 + (64/9\pi^2)\omega_e^4 b_{NL}^2]\phi_a{}^4 + [(16/3\pi)b_L b_{NL}\omega_e^3]\phi_a{}^3$$
$$+[(\omega_e^2 - \omega_\phi^2)^2 + b_L^2\omega_e^2]\phi_a{}^2 - \gamma\lambda^2\alpha_m^2\omega_\phi^4 = 0$$

This polynomial of roll amplitude 'ϕ_a' may be used to obtain roll response in frequency domain. It is also possible to estimate second order solution for $\phi_1(t)$ using present method. Since the present work concerns about estimating roll amplitudes rather than time series data, solution up to obtaining this 'frequency-response' relation is considered.

Following are frequency response relations for B2 and B3 type nonlinear equations.

B2 type:

$$\phi_a^{\,6}[k_3^2(9/16)+(1/16)b_{2NL}^2\,\omega_e^2]+[(3/2)(\omega_\phi^2-\omega_e^2)k_3+(1/2)b_{2L}b_{2NL}\,\omega_e^2]\phi_a^{\,4}+$$
$$[(\omega_e^2-\omega_\phi^2)^2+b_{2L}^2\omega_e^2]\phi_a^{\,2}-\gamma\lambda^2\alpha_m^2\omega_\phi^4=0$$

B3 type:

$$\phi_a^{\,6}[(9/16)k_3^2+(9/16)b_{3NL}^2\;\omega_e^6]+\phi_a^{\,4}[(3/2)(\omega_\phi^2-\omega_e^2)k_3+(3/2)\,b_{3L}b_{3NL}\,\omega_e^4]+$$
$$\phi_a^{\,2}[(\omega_e^2-\omega_\phi^2)+b_{3L}^2\omega_e^2]-\gamma\lambda^2\alpha_m^2\omega_\phi^4=0$$

All the parameters in above polynomial are known except roll amplitude 'ϕ_a' which can be solved by using root finding algorithm. Roll amplitudes obtained based on real roots of polynomial.

Fig.3 shows analytical maximum roll amplitudes at wave frequency equal to natural roll frequency of ship. Fig.4 shows difference between maximum roll amplitude calculated and experiment data. X-axis shows wave height at full scale.

6. Results and Discussions

Fig.1 shows roll response curve obtained from experiments corresponding to full load condition at three different wave heights of 0.511m, 0.681m and 0.851m at full scale. Ratio of wave height to draft is 0.17, 0.22 and 0.28 respectively for these wave heights. Though experiments are conducted with heave free and heave fix arrangements, much difference is not found in maximum roll amplitudes. Fig.2 shows non-dimensionalised roll response. It may be observed from these graphs the influence of non-linearities at resonance even at relatively low wave slope experiments. Experiment results clearly show the influence of wave slope on maximum roll amplitude especially at resonance frequency. Nonlinearities may be more prominent if experiments are conducted for higher wave slopes. Experiment clearly shows dependency of roll response on wave slope that is height and length of waves the ship encounters.

Fig. 3 shows only maximum roll amplitudes at resonance obtained with analytical solution of non-linear equation. Non-linear equations were solved by use of innovate classical perturbation technique. Analytical maximum amplitudes at resonance also show importance of wave slope to determine roll amplitude. The slope of curve is prominent even in rage of relatively lower wave slopes where experiments are carried out for comparison. It may be observed that 'frequency-response' relation takes into account wave slope, which is an important factor in roll response as it may be observed from both experiment and analytical results.

Fig.4 shows deviation of calculated maximum roll amplitude from experiments at resonance. Negative values indicated when calculated value is less than experiment value. Maximum roll resonance amplitude obtained with B2 and B3 type damping is almost same though nonlinear damping coefficients in respective equations differs very much (Table 2). Maximum roll amplitudes obtained with B2 and B3 non-linear roll equations are comparable with experiment results. Difference between maximum roll amplitude obtained analytically with B1 type and experiments are relatively high.

7. CONCLUSIONS

Pure transverse roll experiments in regular waves have been conducted on a model of training ship by varying wave frequency at three different wave height. Experiments are conducted with and without restrain of heave motion. . Much difference is not found between maximum roll amplitude obtained with and without of heave restrainment. Forced roll experiments carried out in present study indicate the importance of nonlinearities at resonance. Thus it is important to take into account nonlinear damping and restoring moment in estimating the maximum roll amplitude at resonance. Both experiment and analytical results indicates importance of wave slope in estimation of roll amplitude. Roll amplitude increase as wave slope increases. The present study shows the influence of wave slope on maximum roll amplitude. Estimated maximum roll amplitude with 'angle dependent' and 'cubic' damping terms gives better results than 'quadratic damping ' with presented analytical method.

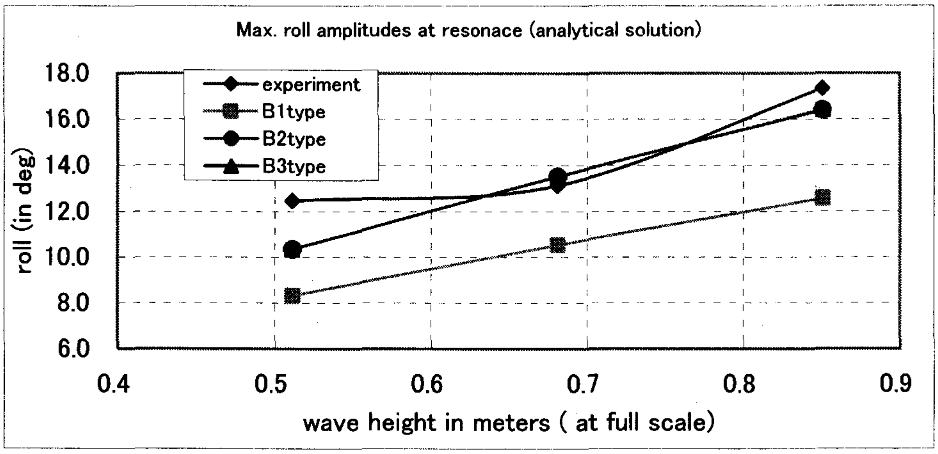

Fig.3 Maximum roll amplitudes (analytical calculations)

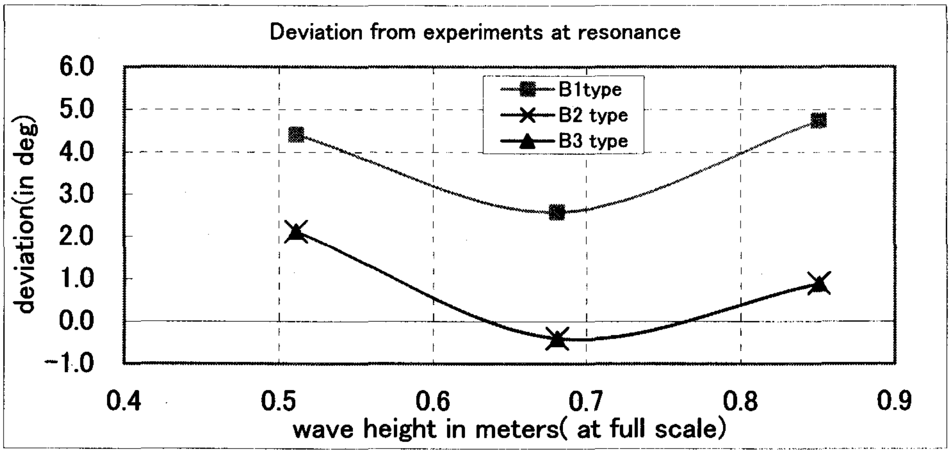

Fig.4 Difference from experiments (analytical calculations)(-ve values indicates when calculations are higher than experiments)

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