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Inference of the Exponential Distribution Based on Multiply Type-II Censored Samples

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Abstract

In this paper, we derive the approximate maximum likelihood estimators of the scale parameter and location parameter of the exponential distribution based on multiply Type-II censored samples. Then three type tests, including the modified Cramer-von Mises test, the modified Watson test and the modified Kolmogorov-Smirnov test are developed for the exponential distribution based on multiply Type-II censored samples by using the proposed estimators.

For each test, Monte Carlo techniques are used to generate critical values. The powers of these tests are investigated under several alternative distributions.

Keywords : Approximate maximum likelihood estimator, Cramer-von Mises test, Exponential distribution, Kolmogorov-Smirnov test, Multiply Type-II censored sample, Watson test.

1. Introduction

The exponential distribution occupies an important position in life testing and reliability problems, especially in the area of industrial life testing. Even when the simple mathematical form of the distribution is inadequate to describe real-life complexity, it often serves as a bench-mark with reference to which effects of departures to allow for specific types of disturbance can be assessed. The failure time X is said to follow two-parameter exponential distribution if the probability density function (pdf) of X is of the form

$$f(x;\theta,\sigma)=\frac{1}{\sigma}\exp\left(-\frac{x-\theta}{\sigma}\right),\quad x>\theta,\ \sigma>0 \tag{1.1}$$

and the cumulative distribution function (cdf)

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$$F(x;\theta,\sigma)=1-\exp\left(-\frac{x-\theta}{\sigma}\right),\quad x>\theta,\ \sigma>0. \tag{1.2}$$

The data for estimating the scale and the location parameters of the two-parameter exponential distribution is usually obtained through Type-II censored sampling scheme. The problem of estimating parameters have been investigated by many authors.

Especially, the approximate maximum likelihood estimating method was first developed by Balakrishnan (1989) for the purpose of providing the explicit estimators of the scale parameter in the Rayleigh distribution.

Kang and Cho (1997) studied estimation for the exponential distribution under general progressive Type-II censored samples. Childs et al. (2003) discussed the exact likelihood inference based on Type-I and Type-II hybrid censored samples from the exponential distribution. Lin and Balakrishnan (2003) developed the exact prediction intervals for the failure times from the one-parameter and two- parameter exponential distributions based on doubly Type-II censored samples and they presented a computational algorithm for the determination of the exact percentage points of the pivotal quantities used in the construction of the prediction intervals.

Multiply Type-II censoring is a more general than Type-II censoring, but mathematically and numerically much more complicated censoring scheme. Balasubramanian and Balakrishnan (1992), Upadhyay et al. (1996) considered estimation for the exponential distribution under multiply Type-II censoring. Recently, Kang (2003) proposed the AMLE of the location and the scale parameters of the two-parameter exponential distribution with multiply Type-II censoring.

Porter III et al. (1992) developed three modified Kolmogorov-Smirnov, Anderson-Darling, and Cramer-von Mises tests for the Pareto distribution with unknown parameters of the location and scale and known shape parameter based on the complete samples. Puig and Stephens (2000) studied some tests of fit for the Laplace distribution based on the EDF statistics and the application of the Laplace distribution in the least absolute deviations (LAD) regression.

In this paper, we consider unbiased estimator of the location parameter, and we derive estimator of the location parameter by minimizing the mean squared error (MSE) of the linear combination of some available order statistics. we also propose the AMLEs of the scale parameter σ of the two-parameter exponential distribution with multiply Type-II censoring. The scale parameter is estimated by approximate maximum likelihood estimation method using two different types Taylor series expansions. Then three type tests, including the modified Cramer-von Mises test, the modified Watson test and the modified Kolmogorov-Smirnov test are developed for the exponential distribution based on multiply Type-II censored samples by using the proposed estimators.

2. Approximate Maximum Likelihood Estimators

Let

$$X_{a_1:n} \le X_{a_2:n} \le \cdots \le X_{a_s:n} \tag{2.1}$$

be the available multiply Type-II censored sample from the exponential distribution with pdf (1.1), where $1 \le a_1 < a_2 < \cdots < a_s \le n$.

Let $a_0 = 0$, $a_{s+1} = n+1$, $F(x_{a_0:n}) = 0$, $F(x_{a_{s+1}:n}) = 1$, then the likelihood function based on the multiply Type-II censored sample (2.1) is given by

$$\begin{aligned} L &= \frac{n!}{\prod_{j=1}^{s+1}(a_j - a_{j-1} - 1)!} \prod_{j=1}^{s+1}[F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})]^{a_j - a_{j-1} - 1} \frac{1}{\sigma^s} \prod_{j=1}^{s} f(Z_{a_j:n}) \\ &= \frac{1}{\sigma^s} \frac{n!}{\prod_{j=1}^{s+1}(a_j - a_{j-1} - 1)!} [F(Z_{a_1:n})]^{a_1 - 1} [1 - F(Z_{a_s:n})]^{n - a_s} \\ &\times \prod_{j=1}^{s} f(Z_{a_j:n}) \prod_{j=2}^{s}[F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})]^{a_j - a_{j-1} - 1} \end{aligned} \tag{2.2}$$

where $Z_{i:n} = (X_{i:n} - \theta)/\sigma$, and $f(z) = e^{-z}$ and $F(z) = 1 - e^{-z}$ are pdf and cdf of the standard exponential distribution, respectively.

2.1 Estimation of the location parameter

We consider some estimators of the location parameter θ. The following estimator is well known estimator of the location parameter.

$$\widehat{\theta_1} = X_{a_1:n}. \tag{2.2.1}$$

But $\widehat{\theta_1}$ always overestimate the location parameter θ, so we consider another unbiased estimator which is linear combination of the minimum available order statistic as follows;

$$\widehat{\theta_2} = c_1 X_{a_1:n} + c_2 X_{a_2:n}. \tag{2.1.2}$$

The expectation of $\widehat{\theta_2}$ is given by

$$E(\widehat{\theta_2}) = (c_1 + c_2)\theta + \sigma\left[c_1 \sum_{j=1}^{a_1}(n-j+1)^{-1} + c_2 \sum_{j=1}^{a_2}(n-j+1)^{-1}\right] \tag{2.1.3}$$

where c_1 and c_2 are constants.

From equations (2.1.2) and (2.1.3), we can easily obtain an unbiased estimator of the location parameter as follows;

$$\widehat{\theta_2} = \frac{1}{h(a_2)-h(a_1)}\left[h(a_2)X_{a_1:n} - h(a_1)X_{a_2:n}\right] \tag{2.1.4}$$

where $h(a) = \sum_{j=1}^{a}(n-j+1)^{-1}$.

Also we can derive the other estimator by minimizing the MSE among the class of estimators of the form $[1-(s-1)d]X_{a_1:n} + d\sum_{j=2}^{s}X_{a_j:n}$ where d is constant.

Let

$$\widehat{\theta_3} = [1-(s-1)d]X_{a_1:n} + d\sum_{j=2}^{s}X_{a_j:n}. \tag{2.1.5}$$

The MSE of $\widehat{\theta_3}$ is given by

$$\begin{aligned}\text{MSE}(\widehat{\theta_3}) = &\Big[[1-(s-1)d]^2[g(a_1)+h^2(a_1)] \\ &+ d^2\Big\{\sum_{j=2}^{s}g(a_j) + 2\sum_{j=2}^{s-1}(s-j)g(a_j) + \Big(\sum_{j=2}^{s}h(a_j)\Big)^2\Big\} \\ &+ 2d[1-(s-1)d]\Big\{(s-1)g(a_1) + h(a_1)\sum_{j=2}^{s}h(a_j)\Big\}\Big]\sigma^2.\end{aligned} \tag{2.1.6}$$

From equation (2.1.6), we can also obtain the constant d which minimize $\text{MSE}(\widehat{\theta_3})$ by differentiation. So we propose the estimator of the location parameter as follows;

$$\widehat{\theta_3} = [1-(s-1)d]X_{a_1:n} + d\sum_{j=2}^{s}X_{a_j:n} \tag{2.1.7}$$

where

$$d = \frac{h(a_1)\left[(s-1)h(a_1) - \sum_{j=2}^{s}h(a_j)\right]}{v}$$

$$\begin{aligned}v = &(s-1)^2[h^2(a_1)-g(a_1)] + \sum_{j=2}^{s}g(a_j) + 2\sum_{j=1}^{s-1}(s-j)g(a_j) + \Big[\sum_{j=2}^{s}h(a_j)\Big]^2 \\ &-2(s-1)h(a_1)\sum_{j=2}^{s}h(a_j)\end{aligned}$$

and

$$g(a) = \sum_{j=1}^{a}(n-j+1)^{-2}.$$

2.2 Estimation of the scale parameter

We consider the estimation of the scale parameter σ. From equation (2.2), we can obtain the following likelihood equation for σ.

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{1}{\sigma}\Bigg[s+(a_1-1)\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})} Z_{a_1:n} - (n-a_s)\frac{f(Z_{a_s:n})}{1-F(Z_{a_s:n})} Z_{a_s:n} - \sum_{j=1}^{s} Z_{a_j:n} + \sum_{j=2}^{s}(a_j-a_{j-1}-1)\frac{f(Z_{a_j:n})Z_{a_j:n} - f(Z_{a_{j-1}:n})Z_{a_{j-1}:n}}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})}\Bigg] = 0. \tag{2.2.1}$$

The equation (2.2.1) does not admit an explicit solution for σ. But we can expand the following functions

$$\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})},\quad \frac{f(Z_{a_j:n})}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})},\quad \frac{f(Z_{a_{j-1}:n})}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})}$$

in Taylor series around the points ξ_{a_1} and $(\xi_{a_j}, \xi_{a_{j-1}})$ respectively, where

$$\xi_{a_1} = F^{-1}(p_{a_1}) = -\ln(1-p_{a_1})$$
$$\xi_{a_j} = F^{-1}(p_{a_j}) = -\ln(1-p_{a_j})$$
$$\xi_{a_{j-1}} = F^{-1}(p_{a_{j-1}}) = -\ln(1-p_{a_{j-1}})$$

and

$$p_i = \frac{i}{n+1}.$$

Therefore, we can approximate these functions by

$$\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})} \simeq \alpha_1 + \beta_1 Z_{a_1:n} \tag{2.2.2}$$

$$\frac{f(Z_{a_j:n})}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})} \simeq \alpha_{1j} + \beta_{1j} Z_{a_j:n} + \gamma_{1j} Z_{a_{j-1}:n} \tag{2.2.3}$$

and

$$\frac{f(Z_{a_{j-1}:n})}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})} \simeq \alpha_{2j} + \beta_{2j} Z_{a_j:n} + \gamma_{2j} Z_{a_{j-1}:n} \tag{2.2.4}$$

where

$$\alpha_1 = \frac{f(\xi_{a_1})}{p_{a_1}}\left[1+\xi_{a_1}+\frac{f(\xi_{a_1})}{p_{a_1}}\xi_{a_1}\right]$$

$$\beta_1 = -\frac{f(\xi_{a_1})}{p_{a_1}}\left[1+\frac{f(\xi_{a_1})}{p_{a_1}}\right]$$

$$\alpha_{1j} = \frac{f(\xi_{a_j})}{p_{a_j}-p_{a_{j-1}}}\left(1+\xi_{a_j}+\frac{f(\xi_{a_j})\xi_{a_j}-f(\xi_{a_{j-1}})\xi_{a_{j-1}}}{p_{a_j}-p_{a_{j-1}}}\right)$$

$$\beta_{1j} = -\frac{f(\xi_{a_j})}{p_{a_j}-p_{a_{j-1}}}\left(1+\frac{f(\xi_{a_j})}{p_{a_j}-p_{a_{j-1}}}\right)$$

$$\gamma_{1j} = \frac{f(\xi_{a_j})f(\xi_{a_{j-1}})}{(p_{a_j}-p_{a_{j-1}})^2}$$

$$\alpha_{2j} = \frac{f(\xi_{a_{j-1}})}{p_{a_j}-p_{a_{j-1}}}\left(1+\xi_{a_{j-1}}+\frac{f(\xi_{a_j})\xi_{a_j}-f(\xi_{a_{j-1}})\xi_{a_{j-1}}}{p_{a_j}-p_{a_{j-1}}}\right)$$

$$\beta_{2j} = -\frac{f(\xi_{a_j})f(\xi_{a_{j-1}})}{(p_{a_j}-p_{a_{j-1}})^2} = -\gamma_{1j}$$

and

$$\gamma_{2j} = -\frac{f(\xi_{a_{j-1}})}{p_{a_j}-p_{a_{j-1}}}\left(1-\frac{f(\xi_{a_{j-1}})}{p_{a_j}-p_{a_{j-1}}}\right).$$

By substituting equations (2.2.2), (2.2.3), and (2.2.4) into equation (2.2.1), we obtain the approximate likelihood equation of (2.2.1) as

$$\begin{aligned}\frac{\partial \ln L}{\partial \sigma} &\simeq \frac{\partial \ln L^*}{\partial \sigma}\\ &= -\frac{1}{\sigma}\Big[s+(a_1-1)(\alpha_1+\beta_1 Z_{a_1:n})Z_{a_1:n}-(n-a_s)Z_{a_s:n}\\ &\quad+\sum_{j=2}^{s}(a_j-a_{j-1}-1)[(\alpha_{1j}+\beta_{1j}Z_{a_j:n}+\gamma_{1j}Z_{a_{j-1}:n})Z_{a_j:n}\\ &\quad-(\alpha_{2j}+\beta_{2j}Z_{a_j:n}+\gamma_{2j}Z_{a_{j-1}:n})Z_{a_{j-1}:n}]-\sum_{j=1}^{s}Z_{a_j:n}\Big]\\ &=0\,.\end{aligned} \tag{2.2.5}$$

Equation (2.2.5) is quadratic in σ as follows;

$$s\sigma^2 + B_{1i}\sigma + C_{1i} = 0 \tag{2.2.6}$$

where

$$\begin{aligned}B_{1i} =\ & (a_1-1)\alpha_1 X_{a_1:n}-(n-a_s)X_{a_s:n}-\sum_{j=1}^{s}X_{a_j:n}\\ &+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\alpha_{1j}X_{a_j:n}-\alpha_{2j}X_{a_{j-1}:n})\\ &-\Big[(a_1-1)\alpha_1-(n-a_s)-s+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\alpha_{1j}-\alpha_{2j})\Big]\widehat{\theta_i}\end{aligned}$$

$$C_{1i}=\sum_{j=2}^{s}(a_j-a_{j-1}-1)\{\beta_{1j}(X_{a_j:n}-\widehat{\theta_i})^2+2\gamma_{1j}(X_{a_j:n}-\widehat{\theta_i})(X_{a_{j-1}:n}-\widehat{\theta_i})$$
$$-\gamma_{2j}(X_{a_{j-1}:n}-\widehat{\theta_i})^2\}+(a_1-1)\beta_1(X_{a_1:n}-\widehat{\theta_i})^2.$$

and $\widehat{\theta_0}=\theta_0$ is known location parameter.

Upon solving equation (2.2.6) for σ, we first derive an AMLE of σ as

$$\widehat{\sigma}_{1i}=\frac{-B_{1i}+\sqrt{B_{1i}^{2}-4sC_{1i}}}{2s},\quad i=0,1,2,3. \tag{2.2.7}$$

Second, we can expand the functions.

$$\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})}Z_{a_1:n}\text{ and }\frac{f(Z_{a_j:n})Z_{a_j:n}-f(Z_{a_{j-1}:n})Z_{a_{j-1}:n}}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})}$$

Therefore, we can approximate these functions by

$$\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})}Z_{a_1:n}\simeq\alpha_2+\beta_2Z_{a_1:n} \tag{2.2.8}$$

and

$$\frac{f(Z_{a_j:n})Z_{a_j:n}-f(Z_{a_{j-1}:n})Z_{a_{j-1}:n}}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})}\simeq\alpha_j+\beta_jZ_{a_j:n}+\gamma_jZ_{a_{j-1}:n} \tag{2.2.9}$$

where

$$\alpha_2=\frac{f(\xi_{a_1})}{p_{a_1}}\xi_{a_1}\left[\frac{f(\xi_{a_1})}{p_{a_1}}\xi_{a_1}+\xi_{a_1}\right]$$

$$\beta_2=\frac{f(\xi_{a_1})}{p_{a_1}}\left[1-\xi_{a_1}-\frac{f(\xi_{a_1})}{p_{a_1}}\xi_{a_1}\right]$$

$$\alpha_j=\frac{f(\xi_{a_j})\xi_{a_j}^{2}-f(\xi_{a_{j-1}})\xi_{a_{j-1}}^{2}}{p_{a_j}-p_{a_{j-1}}}+\left(\frac{f(\xi_{a_j})\xi_{a_j}-f(\xi_{a_{j-1}})\xi_{a_{j-1}}}{p_{a_j}-p_{a_{j-1}}}\right)^2$$

$$\beta_j=\frac{f(\xi_{a_j})}{p_{a_j}-p_{a_{j-1}}}\left(1-\xi_{a_j}-\frac{f(\xi_{a_j})\xi_{a_j}-f(\xi_{a_{j-1}})\xi_{a_{j-1}}}{p_{a_j}-p_{a_{j-1}}}\right)$$

and

$$\gamma_j=-\frac{f(\xi_{a_{j-1}})}{p_{a_j}-p_{a_{j-1}}}\left(1-\xi_{a_{j-1}}-\frac{f(\xi_{a_j})\xi_{a_j}-f(\xi_{a_{j-1}})\xi_{a_{j-1}}}{p_{a_j}-p_{a_{j-1}}}\right)$$

in Taylor series around the points ξ_{a_1} and $(\xi_{a_j},\ \xi_{a_{j-1}})$ respectively.

By substituting equations (2.2.8) and (2.2.9) into equation (2.2.1), we obtain the secondly approximate likelihood equation of (2.2.1) as

$$\frac{\partial \ln L}{\partial \sigma} \simeq \frac{\partial \ln L^*}{\partial \sigma}$$
$$= -\frac{1}{\sigma}\Big[s + (a_1 - 1)(\alpha_2 + \beta_2 Z_{a_1:n}) - (n - a_s) Z_{a_s:n}$$
$$+ \sum_{j=2}^{s} (a_j - a_{j-1} - 1)(\alpha_j + \beta_j Z_{a_j:n} + \gamma_j Z_{a_{j-1}:n}) - \sum_{j=1}^{s} Z_{a_j:n} \Big] \qquad (2.2.10)$$
$$= 0.$$

From equation (2.2.10), we can derive more simple estimator of σ which is linear function of the available order statistics as

$$\hat{\sigma}_{2i} = -\frac{B_{2i}}{A_2} \qquad (2.2.11)$$

where

$$A_2 = s + (a_1 - 1)\alpha_2 + \sum_{j=2}^{s} (a_j - a_{j-1} - 1)\alpha_j$$

and

$$B_{2i} = (a_1 - 1)\beta_2 X_{a_1:n} - (n - a_s) X_{a_s:n} - \sum_{j=1}^{s} X_{a_j:n}$$
$$+ \sum_{j=2}^{s} (a_j - a_{j-1} - 1)(\beta_j X_{a_j:n} + \gamma_j X_{a_{j-1}:n})$$
$$- \Big[(a_1 - 1)\beta_2 - (n - a_s) - s + \sum_{j=2}^{s} (a_j - a_{j-1} - 1)(\beta_j + \gamma_j) \Big] \hat{\theta}_i .$$

Third, Johnson et al. (1994) derived the best linear unbiased estimator of the scale parameter σ when the location parameter θ is known as follows;

$$\hat{\sigma} = \Big[\sum_{j=1}^{k} \Big(\frac{w_{1j}}{w_{2j}} - \frac{w_{1,j+1}}{w_{2,j+1}} \Big) X_{a_j:n} - \frac{w_{11}}{w_{21}} \theta \Big] \Big[\sum_{j=1}^{k} \frac{w_{1j}}{w_{2j}} \Big]^{-1}$$

where

$$w_{mj} = \sum_{i=a_{j-1}}^{a_j - 1} (n - i)^{-m}$$

and $\frac{w_{10}}{w_{20}}$ and $\frac{w_{1,k+1}}{w_{2,k+1}}$ defined to be zero.

We can use the estimator $\hat{\sigma}$ when the location parameter θ is unknown as follows;

$$\hat{\sigma}_{3i} = \Big[\sum_{j=1}^{k} \Big(\frac{w_{1j}}{w_{2j}} - \frac{w_{1,j+1}}{w_{2,j+1}} \Big) X_{a_j:n} - \frac{w_{11}}{w_{21}} \hat{\theta}_i \Big] \Big[\sum_{j=1}^{k} \frac{w_{1j}}{w_{2j}} \Big]^{-1} . \qquad (2.2.12)$$

2.3 Goodness of fit tests

Suppose a given random sample of size n is $X_1, X_2, \ldots, X_n$ and let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be the order statistics. Further that the distribution function of X is $F(x)$. The empirical distribution function (EDF) $F_n(x)$ is defined by

$$F_n(x) = \frac{\#[x_i \le x]}{n}, \quad -\infty \le x \le \infty \tag{2.3.1}$$

The Kolmogorov-Smirnov statistics are defined by

$$\begin{aligned} D^+ &= \sup_x [F_n(x) - F(x)] \\ D^- &= \sup_x [F(x) - F_n(x)] \\ D &= \max[D^+, D^-], \end{aligned} \tag{2.3.2}$$

and the Cramer-von Mises statistic is generally defined to be the statistic

$$W^2 = n\int_{-\infty}^{\infty} [F_n(x) - F(x)]^2 dF(x) \tag{2.3.3}$$

and the Anderson-Darling statistic is defined by

$$A^2 = n\int_{-\infty}^{\infty} [F_n(x) - F(x)]^2 \psi(x)\, dF(x), \tag{2.3.4}$$

where $\psi(x) = F(x)[1 - F(x)]^{-1}$.

Let $U_i = F(X_i)$ then $U_i \sim U(0,1)$. But if $F(x)$ contains unknown parameters, we can denoted the cdf $F(x)$ by $F(x; \theta_1, \ldots \theta_n)$ and we must estimate the parameters from the sample and the estimated values are used in $F(x; \theta_1, \ldots \theta_n)$ to make the transformation $U_{i:n} = F(X_{i:n}; \theta_1, \ldots \theta_n)$, for $i = 1, \ldots, n$. The $u_{i;n}$ will be in ascending order.

Then the Kolmogorov-Smirnov statistics are computed from

$$\begin{aligned} D^+ &= \max_{1 \le i \le n}\left[\frac{i}{n} - u_{i:n}\right] \\ D^- &= \max_{1 \le i \le n}\left[u_{i:n} - \frac{i}{n}\right] \\ D &= \max[D^+, D^-], \end{aligned} \tag{2.3.5}$$

and the Anderson-Darling statistic is computed from

$$A^2 = -n - \frac{1}{n}\left(\sum_{i=1}^{n}(2i-1)\left[\ln u_{i:n} + \ln(1-u_{n+1-i:n})\right]\right), \tag{2.3.6}$$

and the Cramer-von Mises statistic is computed from

$$W^2 = \sum_{i=1}^{n}\left(u_{i:n} - \frac{2i-1}{2n}\right)^2 + \frac{1}{12n}. \tag{2.3.7}$$

Porter III et. al. (1992) developed the modified EDF type tests that are the Kolmogorov-Smirnov test, Anderson-Darling test, and Cramer-von Mises test for complete samples. But these tests can't use for the censored samples. So we propose the modified Kolmogorov-Smirnov test, Anderson-Darling test, and Cramer-von Mises test by using the proposed AMLEs $\hat{\sigma}_{1i}$ and $\hat{\sigma}_{2i}$ that can be used for multiply Type-II censored samples.

The AMLEs $\hat{\sigma}_{kl}$ and $\hat{\theta}_l$ were used to find the hypothesized cdf as follows;

$$P_{a_j,k,l} = F(x_{a_j:n}, \hat{\sigma}_{kl}, \hat{\theta}_l), \quad j=1,\ldots,s, \ k=1,2, \ l=0,1,2.$$

Then we propose three modified test statistics based on multiply Type-II censored samples as follows;

The modified Kolmogorov-Smirnov test statistic:

$$D^+_{k,l} = \max_{1\le a_j\le n}\left[\frac{a_j}{s} - P_{a_j,k,l}\right]$$

$$D^-_{k,l} = \max_{1\le a_j\le n}\left[P_{a_j,k,l} - \frac{a_j}{s}\right]$$

$$D_{k,l} = \max\left[D^+_{k,l},\ D^-_{k,l}\right]. \tag{2.3.8}$$

The modified Anderson-Darling test statistic:

$$A^2_{k,l} = -s - \frac{1}{s}\sum_{j=1}^{s}(2a_j-1)\left[\ln P_{a_j,k,l} + \ln(1-P_{a_s+1-a_j,k,l})\right]. \tag{2.3.9}$$

The modified Cramer-von Mises test statistic:

$$W_{k,l}{}^2 = \frac{1}{12s} + \sum_{j=1}^{s}\left(P_{a_j,k,l} - \frac{2a_j-1}{2s}\right)^2. \tag{2.3.10}$$

This procedure was repeated 10,000 times for all three tests, each sample size $n=20, 50$, and $k=1,2$, and $l=0,2,3$. A power study was made between the modified Kolmogorov-Smirnov, modified Anderson-Darling, and modified Cramer-von Mises goodness-of-fit tests for the two-parameter exponential distribution based on multiply Type-II censored samples.

From equations (2.1.1), (2.1.4), (2.1.7), (2.2.7), (2.2.11), and (2.2.12), the mean squared errors of these estimators are simulated by Monte Carlo method for sample size $n=20, 50$ and various choices of censoring. The simulation procedure is repeated 10,000 times in multiply Type-II censored samples. From Table 1, the estimators $\widehat{\theta_3}$ is more efficient than the other estimators in the sense of MSE. From Table 2, when the location parameter is known, the estimators $\widehat{\sigma}_{20}$ and $\widehat{\sigma}_{30}$ of the scale parameter are generally more efficient than the estimator $\widehat{\sigma}_{10}$ in the sense of MSE. When the location parameter is unknown, the estimators $\widehat{\sigma}_{1i}$, $\widehat{\sigma}_{2i}$, and $\widehat{\sigma}_{3i}$ are generally efficient with using the estimator $\widehat{\theta_1}$ of the location parameter θ.

From Table 3, when the alternative distribution is beta distributions, the modified Cramer-von Mises test is more powerful for the complete sample or known location parameter. For lognormal alternative distribution, all tests are no good but the modified Cramer-von Mises test is generally a little more powerful than the other tests. For normal alternative distribution, the powers of three tests are very good when the location parameter is known, and the modified Cramer-von Mises test is more powerful for the complete samples. As expected, the power is generally increase as k decreases and sample size n increases.

Table 1. The relative mean squared errors for the estimators of the location parameter θ.

n	k	a_j	MSE $\widehat{\theta_1}$	MSE $\widehat{\theta_2}$	MSE $\widehat{\theta_3}$	n	k	a_j	MSE $\widehat{\theta_1}$	MSE $\widehat{\theta_2}$	MSE $\widehat{\theta_3}$
20	0	1~20	0.0048	0.0050	0.0025	50	0	1~50	0.0008	0.0008	0.0004
	2	1~18	0.0048	0.0050	0.0025		2	1~48	0.0008	0.0008	0.0004
		3~20	0.0334	0.0348	0.0099			3~50	0.0025	0.0024	0.0009
		2~19	0.0158	0.0159	0.0059			2~49	0.0025	0.0024	0.0009
	5	3~17	0.0334	0.0348	0.0103		5	3~47	0.0085	0.0086	0.0019
		4~18	0.0591	0.0604	0.0153			4~48	0.0085	0.0086	0.0019
		2~6 10~19	0.0158	0.0159	0.0059			2~6 10~19 21~50	0.0025	0.0024	0.0009
	6	4~17	0.0591	0.0604	0.0156		6	4~47	0.0025	0.0024	0.0009
		1 2 6~9 12~15 17~20	0.0048	0.0050	0.0025			1 2 6~9 12~15 17~50	0.0008	0.0008	0.0004

Table 2. The relative mean squared errors for the estimators of the scale parameter σ.

n	k	a_j	MSE							
			$\hat{\sigma}_{10}$	$\hat{\sigma}_{11}$	$\hat{\sigma}_{12}$	$\hat{\sigma}_{13}$	$\hat{\sigma}_{20}$	$\hat{\sigma}_{21}$	$\hat{\sigma}_{22}$	$\hat{\sigma}_{23}$
20	0	1~20	0.0508	0.0505	0.0535	0.0532	0.0508	0.0505	0.0535	0.0532
	2	1~18	0.0567	0.0562	0.0600	0.0596	0.0567	0.0562	0.0600	0.0596
		3~20	0.0556	0.0565	0.0602	0.0593	0.0508	0.0653	0.0602	0.0593
		2~19	0.0577	0.0565	0.0602	0.0595	0.0537	0.0588	0.0602	0.0595
	5	3~17	0.0661	0.0677	0.0734	0.0717	0.0599	0.0797	0.0734	0.0717
		4~18	0.0629	0.0680	0.0737	0.0717	0.0567	0.0926	0.0737	0.0716
		2~6 10~19	0.0651	0.0599	0.0664	0.0655	0.0537	0.0588	0.0603	0.0596
	6	4~17	0.0666	0.0725	0.0793	0.0767	0.0599	0.1002	0.0792	0.0767
		1 2 6~9 12~15 17~20	0.0705	0.0630	0.0734	0.0728	0.0511	0.0507	0.0538	0.0536
50	0	1~50	0.0196	0.0196	0.0199	0.0199	0.0196	0.0196	0.0199	0.0199
	2	1~48	0.0208	0.0204	0.0207	0.0207	0.0208	0.0204	0.0207	0.0207
		3~50	0.0203	0.0205	0.0208	0.0208	0.0196	0.0222	0.0208	0.0208
		2~49	0.0207	0.0204	0.0207	0.0207	0.0199	0.0209	0.0207	0.0207
	5	3~47	0.0214	0.0218	0.0222	0.0221	0.0203	0.0238	0.0222	0.0221
		4~48	0.0219	0.0218	0.0221	0.0221	0.0208	0.0259	0.0221	0.0221
		2~6 10~19 21~50	0.0225	0.0213	0.0224	0.0223	0.0196	0.0206	0.0204	0.0204
	6	4~47	0.0226	0.0223	0.0227	0.0226	0.0196	0.0266	0.0227	0.0226
		1 2 6~9 12~15 17~50	0.0228	0.0216	0.0232	0.0231	0.0196	0.0196	0.0199	0.0199

n	k	a_j	MSE			
			$\hat{\sigma}_{30}$	$\hat{\sigma}_{31}$	$\hat{\sigma}_{32}$	$\hat{\sigma}_{33}$
20	0	1~20	0.0506	0.0503	0.0531	0.0529
	2	1~18	0.0568	0.0566	0.0601	0.0597
		3~20	0.0506	0.0651	0.0595	0.0586
		2~19	0.0531	0.0584	0.0595	0.0589
	5	3~17	0.0605	0.0803	0.0735	0.0719
		4~18	0.0569	0.0930	0.0735	0.0715
		2~6 10~19	0.0534	0.0586	0.0598	0.0592
	6	4~17	0.0605	0.1008	0.0796	0.0772
		1 2 6~9 12~15 17~20	0.0506	0.0504	0.0532	0.0529
50	0	1~50	0.0200	0.0200	0.0204	0.0204
	2	1~48	0.0208	0.0208	0.0213	0.0213
		3~50	0.0200	0.0224	0.0213	0.0213
		2~49	0.0203	0.0213	0.0214	0.0213
	5	3~47	0.0212	0.0239	0.0227	0.0226
		4~48	0.0208	0.0260	0.0228	0.0227
		2~6 10~19 21~50	0.0200	0.0209	0.0210	0.0209
	6	4~47	0.0212	0.0266	0.0233	0.0232
		1 2 6~9 12~15 17~50	0.0200	0.0200	0.0204	0.0204

Table 3. The powers of the modified EDF types of tests for several alternative distributions.

n	k	a_j	The modified Kolmogorov-Smirnov test (Beta(3, 2))					
			$\hat{\sigma}_{10}$	$\hat{\sigma}_{12}$	$\hat{\sigma}_{13}$	$\hat{\sigma}_{20}$	$\hat{\sigma}_{22}$	$\hat{\sigma}_{23}$
20	0	1~20	0.9938	0.6983	0.7007	0.9938	0.6983	0.7007
	2	1~18	0.9209	0.3877	0.3143	0.9209	0.3877	0.3143
		2~19	0.9974	0.3967	0.2707	0.9704	0.2888	0.2697
	5	1 2 18 19 20	0.9974	0.3051	0.2665	0.9826	0.2968	0.2651
50	0	1~50	1.0000	0.9990	0.9989	1.0000	0.9990	0.9989
	2	1~48	1.0000	0.9923	0.9928	1.0000	0.9923	0.9928
		2~49	1.0000	0.9728	0.9827	1.0000	0.9740	0.9827
	5	2~6 10~19 21~50	1.0000	0.9974	0.9986	1.0000	0.9988	0.9991

n	k	a_j	The modified Cramer-von Mises test (Beta(3, 2))					
			$\hat{\sigma}_{10}$	$\hat{\sigma}_{12}$	$\hat{\sigma}_{13}$	$\hat{\sigma}_{20}$	$\hat{\sigma}_{22}$	$\hat{\sigma}_{23}$
20	0	1~20	1.0000	0.9000	0.9337	1.0000	0.9000	0.9337
	2	1~18	0.9953	0.6161	0.5971	0.9953	0.6161	0.5971
		2~19	0.9960	0.3723	0.4351	0.9973	0.4244	0.4355
	5	1 2 18 19 20	0.8803	0.0738	0.0004	0.3308	0.0106	0.0003
50	0	1~50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	2	1~48	1.0000	0.9990	0.9992	1.0000	0.9990	0.9992
		2~49	1.0000	0.9933	0.9984	1.0000	0.9958	0.9984
	5	2~6 10~19 21~50	1.0000	0.9463	0.9591	1.0000	0.9746	0.9749

n	k	a_j	The modified Anderson-Darling test (Beta(3, 2))					
			$\hat{\sigma}_{10}$	$\hat{\sigma}_{12}$	$\hat{\sigma}_{13}$	$\hat{\sigma}_{20}$	$\hat{\sigma}_{22}$	$\hat{\sigma}_{23}$
20	0	1~20	1.0000	0.8362	0.9311	1.0000	0.8362	0.9311
	2	1~18	0.9966	0.5359	0.6454	0.9966	0.5359	0.6454
		2~19	0.9990	0.2809	0.7416	0.9999	0.3896	0.7421
	5	1 2 18 19 20	0.9510	0.0062	0.2980	0.9856	0.0297	0.3011
50	0	1~50	1.0000	0.9999	1.0000	1.0000	0.9999	1.0000
	2	1~48	1.0000	0.9992	0.9997	1.0000	0.9992	0.9997
		2~49	1.0000	0.9964	0.9999	1.0000	0.9967	0.9999
	5	2~6 10~19 21~50	1.0000	0.9993	0.9998	1.0000	1.0000	1.0000

n	k	a_j	The modified Kolmogorov-Smirnov test (N(5, 1))					
			$\hat{\sigma}_{10}$	$\hat{\sigma}_{12}$	$\hat{\sigma}_{13}$	$\hat{\sigma}_{20}$	$\hat{\sigma}_{22}$	$\hat{\sigma}_{23}$
20	0	1~20	1.0000	0.6985	0.6803	1.0000	0.6985	0.6803
	2	1~18	1.0000	0.4079	0.3024	1.0000	0.4079	0.3024
		2~19	1.0000	0.3491	0.1439	1.0000	0.2193	0.1435
	5	1 2 18 19 20	1.0000	0.2968	0.1915	1.0000	0.2484	0.1872
50	0	1~50	1.0000	0.9971	0.9975	1.0000	0.9971	0.9975
	2	1~48	1.0000	0.9869	0.9863	1.0000	0.9869	0.9863
		2~49	1.0000	0.9445	0.9530	1.0000	0.9497	0.9530
	5	2~6 10~19 21~50	1.0000	0.9557	0.9564	1.0000	0.9570	0.9535

Table 3. (continued)

n	k	a_j	The modified Cramer-von Mises test (N(5, 1))					
			$\hat{\sigma}_{10}$	$\hat{\sigma}_{12}$	$\hat{\sigma}_{13}$	$\hat{\sigma}_{20}$	$\hat{\sigma}_{22}$	$\hat{\sigma}_{23}$
20	0	1~20	1.0000	0.8747	0.8953	1.0000	0.8747	0.8953
	2	1~18	1.0000	0.6589	0.6400	1.0000	0.6589	0.6400
		2~19	1.0000	0.3559	0.3612	1.0000	0.4009	0.3615
	5	1 2 18 19 20	1.0000	0.0733	0.0003	0.9954	0.0095	0.0003
50	0	1~50	1.0000	0.9998	0.9999	1.0000	0.9998	0.9999
	2	1~48	1.0000	0.9969	0.9979	1.0000	0.9969	0.9979
		2~49	1.0000	0.9840	0.9940	1.0000	0.9875	0.9940
	5	2~6 10~19 21~50	1.0000	0.9228	0.9320	1.0000	0.9552	0.9504

n	k	a_j	The modified Anderson-Darling test (N(5, 1))					
			$\hat{\sigma}_{10}$	$\hat{\sigma}_{12}$	$\hat{\sigma}_{13}$	$\hat{\sigma}_{20}$	$\hat{\sigma}_{22}$	$\hat{\sigma}_{23}$
20	0	1~20	1.0000	0.8056	0.8835	1.0000	0.8056	0.8835
	2	1~18	1.0000	0.5889	0.6741	1.0000	0.5889	0.6741
		2~19	1.0000	0.2601	0.6718	1.0000	0.3690	0.6723
	5	1 2 18 19 20	1.0000	0.0069	0.2549	1.0000	0.0275	0.2583
50	0	1~50	0.9999	0.9993	0.9999	0.9999	0.9993	0.9999
	2	1~48	0.9999	0.9973	0.9987	0.9999	0.9973	0.9987
		2~49	1.0000	0.9883	0.9983	1.0000	0.9888	0.9983
	5	2~6 10~19 21~50	1.0000	0.9937	0.9977	1.0000	0.9987	0.9998

n	k	a_j	The modified Kolmogorov-Smirnov test (LN(0, 1))					
			$\hat{\sigma}_{10}$	$\hat{\sigma}_{12}$	$\hat{\sigma}_{13}$	$\hat{\sigma}_{20}$	$\hat{\sigma}_{22}$	$\hat{\sigma}_{23}$
20	0	1~20	0.1454	0.1523	0.1743	0.1454	0.1523	0.1743
	2	1~18	0.0877	0.0982	0.1119	0.0877	0.0982	0.1119
		2~19	0.1393	0.1270	0.1584	0.1223	0.1414	0.1583
	5	1 2 18 19 20	0.1209	0.0712	0.1016	0.0948	0.0899	0.1019
50	0	1~50	0.2618	0.2687	0.2823	0.2618	0.2687	0.2823
	2	1~48	0.2048	0.2151	0.2211	0.2048	0.2151	0.2211
		2~49	0.3664	0.2636	0.2815	0.2379	0.2478	0.2815
	5	2~6 10~19 21~50	0.4554	0.3411	0.3555	0.3435	0.3587	0.3717

n	k	a_j	The modified Cramer-von Mises test (LN(0, 1))					
			$\hat{\sigma}_{10}$	$\hat{\sigma}_{12}$	$\hat{\sigma}_{13}$	$\hat{\sigma}_{20}$	$\hat{\sigma}_{22}$	$\hat{\sigma}_{23}$
20	0	1~20	0.1450	0.1378	0.1493	0.1450	0.1378	0.1493
	2	1~18	0.1118	0.1028	0.1127	0.1118	0.1028	0.1127
		2~19	0.1483	0.1317	0.1714	0.1248	0.1321	0.1714
	5	1 2 18 19 20	0.0799	0.0922	0.1130	0.0583	0.1005	0.1123
50	0	1~50	0.2958	0.2574	0.2667	0.2958	0.2574	0.2667
	2	1~48	0.2769	0.2401	0.2430	0.2769	0.2401	0.2430
		2~49	0.4454	0.2814	0.3055	0.3044	0.2685	0.3055
	5	2~6 10~19 21~50	0.5465	0.3997	0.4284	0.4162	0.4203	0.4492

Table 3. (continued)

n	k	a_j	The modified Anderson-Darling test (LN(0, 1))					
			$\hat{\sigma}_{10}$	$\hat{\sigma}_{12}$	$\hat{\sigma}_{13}$	$\hat{\sigma}_{20}$	$\hat{\sigma}_{22}$	$\hat{\sigma}_{23}$
20	0	1~20	0.1349	0.1163	0.1568	0.1349	0.1163	0.1568
	2	1~18	0.0908	0.0735	0.1008	0.0908	0.0735	0.1008
		2~19	0.0758	0.0863	0.1203	0.0781	0.0810	0.1200
	5	1 2 18 19 20	0.0407	0.0634	0.0952	0.0501	0.0628	0.0944
50	0	1~50	0.3393	0.2330	0.2666	0.3393	0.2330	0.2666
	2	1~48	0.2938	0.1824	0.2057	0.2938	0.1824	0.2057
		2~49	0.3162	0.1632	0.2228	0.2525	0.1492	0.2225
	5	2~6 10~19 21~50	0.1016	0.1172	0.1084	0.0853	0.0786	0.0801

References

[1] Balakrishnan, N. (1989). Approximate MLE of the scale parameter of the Rayleigh distribution with censoring, *IEEE Transactions on Reliability,* 38, 355-357.

[2] Balasubramanian, K. and Balakrishnan, N. (1992). Estimation for one-parameter and two-parameter exponential distributions under multiple Type-II censoring, *Statistische Hefte*, 33, 203-216.

[3] Childs, A., Chandrasekar, B., Balakrishnan, N., and Kundu, D. (2003). Exact likelihood inference based on Type-I and Type-II hybrid censored samples from the exponential distribution, *Annals of the Institute of Statistical Mathematics,* 55, 349-330.

[4] Johnson, N. L., Kotz, S., and Balakrishnan, N. (1994). *Continuous Univariate Distributions,* John Wiley & Sons, Inc.

[5] Kang, S. B. (2003). Approximate MLEs for exponential distribution under multiple Type-II censoring, *Journal of the Korean Data & Information Science Society,* 14, 983-988.

[6] Kang, S. B. and Cho, Y. S. (1997). Estimation for exponential distribution under general progressive Type-II censored samples, *Journal of Statistical Theory & Methods,* 8, 239-245.

[7] Lin, C. T. and Balakrishnan, N. (2003). Exact prediction intervals for exponential distributions based on doubly Type-II censored samples, *Journal of Applied Statistics,* 30, 783-801.

[8] Porter III, J, E., Coleman, J. W., and Moore, A. H. (1992). Modified KS, AD, and C-vM tests for the Pareto distribution with unknown location & scale parameters, *IEEE Transaction on Reliability,* 41, 112-117.

[9] Puig, P. and Stephens, M. A. (2000). Tests of fit for the Laplace distribution with applications, *Technometrics,* 42, 417-424.

[10] Upadhyay, S. K., Singh, U., and Shastri, V. (1996). Estimation of exponential parameters under multiply Type-II censoring, *Communications in Statistics- Simulation and Computation,* 25, 801-815.