Singular Value Decomposition Approach to Observability Analysis of GPS/INS

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Abstract

Singular value decomposition (SDV) approach is applied to the observability analysis of GPS/INS in this paper. A measure of observability for a subspace is introduced. It indicates the minimum size of perturbation in the information matrix that makes the subspace unobservable. It is shown that the measure has direct connections with observability of systems, error covariance, and singular structure of the information matrix. The observability measure given in this paper is applicable to the multi-input/multi-output time-varying systems. An example on the observability analysis of GPS/INS is given. The measure of observability is confirmed to be less sensitive to system model perturbation. It is also shown that the estimation error for the vertical component of gyro bias can be considered unobservable for small initial error covariance for a constant velocity horizontal motion.

Keywords: Observability, Measure of observability, Degree of observability, Information matrix, Singular value decomposition (SVD), The global positioning system (GPS), Inertial navigation system (INS).

1. Introduction

Observability analysis of measurement systems has been studied to understand the behavior of error estimator in the aided inertial navigation systems (INS) [1-5]. Analysis of error covariance in the Kalman filter has usually been employed in the empirical studies based on the numerical simulation or test of INS. The error covariance was considered as a useful measure of observability [6]. The error covariance test is also an efficient means of statistical study on the behavior of estimators [7]. However, the empirical method may need analytical study to obtain physical insights on the estimator behavior. Analytic observability study has also been conducted to understand the navigation error behavior in a more systematic way. In the analytic studies the rank of observability matrix was mainly investigated [3,8-12].

In the error covariance test, the size of error covariance matrix is usually considered as the degree of observability in estimation applications. There are a couple of problems in this approach. Both the decrease and decrease rate in the error covariance are very sensitive to the choice of initial error covariance for a given system model. In addition, the error covariance of unobservable states can decrease if cross correlations between the observable states and unobservable states exist in the initial error covariance. On the other hand, in the rank test on the observability matrix, the rank does not provide the degree of observability. It only decides if a system is observable or not. Further more, the rank of a matrix can be sensitive to perturbation.

To study the degree of observability analytically, several measures of observability have been proposed. Frequency domain observability measures for the time invariant systems were suggested in [13,14]. The condition number of the observability matrix for single-output time-varying systems was considered as a measure of observability in [15]. However, these measures are not suitable for the study of time-varying multi-input and multi-output systems. In this paper analysis on observability and estimation is considered for wider class of systems.

A measure of observability for a subspace is introduced to study the degree of observability of the subspace. The measure is defined as the minimum norm of the perturbation in the information matrix that makes the subspace unobservable. This measure can be useful to investigate the estimation of a specific component of state. The relations among the observability measure, system observability, error covariance, and the singular structure of information matrix are investigated with the singular value decomposition (SVD).

The measure of observability of a system can be defined as the minimum observability measure over the whole state-space. Each singular value of the information matrix is the observability measure for the subspace spanned by the corresponding singular vector. Using SVD, the measure of observability for a subspace can be expressed with the combination of singular values. Thus, SVD plays a key role in the observability analysis in this paper.

The measure of observability given in this paper has several advantages over conventional observability test criteria. The measure is defined with information matrix that is determined only by the system model. Information matrix has direct connections with both the error covariance matrix and observability. If the initial state is totally uncertain, the error covariance matrix is the inverse of information matrix. In general, an error covariance matrix is determined by the initial error covariance matrix and information matrix. Unobservable subspace of a system is also the null space of information matrix. Since the SVD is well conditioned to perturbation [16,17], the proposed observability measure is less sensitive to perturbation due to errors in system model or numerical computation. Moreover, the measure is defined for general system model so that it can be applicable to multi-input and multi-output time-varying systems.

There are several useful properties of SVD for observability analysis. The smallest singular value of a matrix is 2-norm distance of the matrix to the set of all rank-deficient matrices [16]. Thus, the smallest singular value of the information matrix can be considered the measure of system observability. With this observability measure, the effect of perturbation in the system model on the system observability can be studied. The largest singular value of a matrix can be used as the size of the matrix. Thus, the magnitude of matrix perturbation is represented by the largest singular value of the perturbation in this paper.

There are several models for the linearized error equations of the inertial navigation systems (INS) mechanization equation.

The choice of the error model depends on the navigation reference frame, error model of navigation sensors, and vehicle motion. Since the navigation equations are non-linear, approximation errors are inevitable in the linearized error models. Usually, small errors are included in accurate navigation systems and relatively large approximation errors can be allowed in the error model of lower grade INS. The problem with the INS error model in the observability analysis is that different results can be obtained for different models. For example, the number of unobservable states for a vehicle that runs at a constant speed in the horizontal plane is three for tactical grade INS [3] and four for very low-grade INS [11]. One of the main motivations of this paper is to provide an observability measure that is less sensitive to perturbation in the system model.

2. Observability Analysis with SVD

In this section a measure of observability and its properties are introduced. The relations among observability, the measure of observability, error covariance, and information matrix are examined with SVD.

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^n$ denote the vector space of real n-vectors, and $\mathbb{R}^{m\times n}$ denotes the set of m-by-n real numbers. For a matrix $M \in \mathbb{R}^{m\times n}$, there exists the singular value decomposition (SVD), $M = UDV^T$, such that $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are orthogonal, $D = diag(\sigma_1, \sigma_2, \cdots, \sigma_p) \in \mathbb{R}^{m\times n}$, $p = \min\{m,n\}$, $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_p \ge 0$. The σ_i are called singular values of M. Columns u_i in U are called left singular vectors of M and columns v_i in V are called right singular vectors of M. $x \sim N(\bar{x}, P)$ denotes the normal random variable with mean $\bar{x}$ and covariance P. $E(x)$ denotes the expected value of x, $\|M\|_2$ denotes the 2-norm of a matrix or vector. Note that if M is a matrix, $\|M\|_2$ is the same of the largest singular value of M. $\bar{\sigma}(\cdot)$ and $\underline{\sigma}(\cdot)$ denote the largest and smallest singular values, respectively.

For the convenience of observability analysis, consider the following estimation problem with noise free plant: Given the measurements $\{y_1, y_2, \cdots, y_k\}$ with $y_i = c_i x_i + v_i$, $x_i = \phi_{i,0} x_0$, $v_i \sim N(0, R_i)$, $x_0 \sim N(\bar{x}_0, P_0)$, $i = 1, 2, \cdots$, $E(v_i v_j^T) = 0$ for $i \ne j$, and $E(v_i x_0^T) = 0$ for all i, find the optimal estimate of x_0, $\bar{x}_0^k$, that minimizes the quadratic form

$$J = \frac{1}{2}\left\{ (x_0 - \bar{x}_0)^T P_0^{-1} (x_0 - \bar{x}_0) + \sum_{i=0}^{k} (y_i - c_i x_i)^T R_i^{-1} (y_i - c_i x_i) \right\} \quad (1)$$

The above weighted-least-squares estimate is identical to the conditional expected-value estimate that is also minimum variance or maximum-likelihood estimate [18,19]. The optimal estimate $\bar{x}_0^k$ is given as

$$\bar{x}_0^k = \left(P_0^{-1} + L_{k,0}\right)^{-1} \left(H_k + P_0^{-1} \bar{x}_0\right) \quad (2)$$

with

$$L_{k,0} = \sum_{i=0}^{k} \phi_{i,0}^T c_i^T R_i^{-1} c_i \phi_{i,0}, \; H_k = \sum_{i=0}^{k} \phi_{i,0}^T c_i^T R_i^{-1} y_i \quad (3)$$

where $L_{k,0}$ is the observability gramian or information matrix. Let $x_0 \in \mathbb{R}^n$. The system is said to be completely observable if and only if $L_{i+n-1,i} > 0$ for all $i \ge 0$ [20]. The system is said to be locally observable at a time step k if the state x_k can be determined from the measurements $\{y_k, y_{k+1}, \cdots, y_{k+n-1}\}$. The system is locally observable at a time step k if and only if $L_{k+n-1,k} > 0$ [15]. Local observability can be useful to investigate observability of systems at a specific time. In the following the properties of local observability is discussed at the time step $k = 0$. If the system is unobservable, then there exists a vector x_u, called an unobservable state, such that $L_{n-1,0} x_u = 0$. The null space of $L_{n-1,0}$ is called the unobservable subspace.

Determination of observability requires rank test on the information matrix. However, the rank test can be sensitive to perturbation in the system model or numerical computation. Furthermore, the rank test does not provide the degree of observability. It only decides if a system is observable or not. In many estimation applications, the degree of observability has been determined from the behavior of error covariance. If the error covariance of a state experiences a large decrease from the initial covariance, the degree of observability for the corresponding state is usually said to be large. In the following, the connections among the information matrix, error covariance matrix, and degree of observability or measure of observability are investigated with SVD techniques.

Let the error covariance matrix be

$$P_0^{n-1} \triangleq E\left[\left(x_0 - \bar{x}_0^{n-1}\right)\left(x_0 - \bar{x}_0^{n-1}\right)^T \right] \quad (4)$$

Then, we have

$$\left(P_0^{n-1}\right)^{-1} = P_0^{-1} + L_{n-1,0} \quad (5)$$

The inverse of error covariance matrix is the sum of the inverse of initial error covariance matrix and information matrix. If we treat the inverse of the initial error covariance as the initial information matrix, then the total information matrix, the sum of the initial information matrix and information matrix, is the same as the inverse of the error covariance matrix, $\left(P_0^{n-1}\right)^{-1}$. Note that $L_{n-1,0}$ contains system model matrices c_i and $\phi_{i,0}$, and that P_0^{-1} is completely independent of system model.

If the measure of observability in estimation problems is assumed to be the opposite concept of the degree of uncertainty, it might be reasonable to define the measure of observability with the inverse of error covariance matrix. Thus, if we want to have a measure of observability that is very closely related to system model, it would be much more desirable to define the measure of observability with $L_{n-1,0}$. In many estimation applications a measure of observability for a subspace can be quite convenient to predict or understand the behavior of a component of the state. For this purpose a measure of observability for a subspace is defined as the minimum norm of matrix perturbation in the information matrix that makes the subspace unobservable. The following norm can be useful to define the observability measure:

$$\mu(M, z) \triangleq \min_{(M-\Delta)z=0} \|\Delta\|_2 \quad (6)$$

where $M, \Delta \in \mathbb{R}^{m\times n}$ and $z \in \mathbb{R}^n$. With the definition, we have the following theorem:

Theorem 1: Let $M \in \mathbb{R}^{m\times n}$, $z \in \mathbb{R}^n$ *with* $\|z\|_2 = 1$. *Then*

$$\mu(M, z) = \sqrt{z^T M^T M z} \quad (7)$$

The proofs of theorems in the paper are omitted and given in [21]. Let UDU^T be the SVD of the information matrix $L_{n-1,0}$ with $D = diag(\sigma_1, \sigma_2, \cdots, \sigma_n)$ and $U = [u_1 \; u_2 \; \cdots \; u_n]$. Then, we have the following measure of observability for a subspace spanned by $z \in \mathbb{R}^n$ with $\|z\|_2 = 1$

$$\mu\left(L_{n-1,0}, z\right) = \sqrt{z^T U D^2 U^T z} \tag{8}$$

In particular, when the subspace is spanned by a singular vector,

$$\mu\left(L_{n-1,0}, u_i\right) = \sigma_i \tag{9}$$

Then the local observability of the system can be determined with the following definition:

$$\underline{\mu}\left(L_{n-1,0}\right) \triangleq \min_{z \in \mathbb{R}^n} \mu\left(L_{n-1,0}, z\right) = \sigma_n \tag{10}$$

Thus if $\underline{\mu}\left(L_{n-1,0}\right) = 0$, then the system is locally unobservable. Therefore a singular value of the information matrix is the measure of observability for the subspace spanned by the corresponding singular vector. A large singular value implies that a large change in the system model is necessary to make the subspace spanned by the corresponding singular vector unobservable. If the initial state is completely unknown such that $P_0^{-1} = 0$, then $P_0^{n-1} = UD^{-1}U^T$, $z^T P_0^{n-1} z = z^T U D^{-1} U^T z$, and $u_i^T P_0^{n-1} u_i = 1/\sigma_i$ with $D^{-1} = diag\left(1/\sigma_1, 1/\sigma_2, \cdots, 1/\sigma_n\right)$. Thus, the error covariance of a state in the subspace spanned by a singular vector is the inverse of the singular value that corresponds to the singular vector. Due to the well-conditioning of singular structure with respect to perturbation, the measure of observability is less sensitive to perturbation due to errors in computation or system model.

In the following relations among observability, error covariance matrix, and information matrix are given. For simplicity of notation, L_m and P_0^m are used for $L_{n-1,0}$ and P_0^{n-1}, respectively, in the following. Since $\left(P_0 - P_0^m\right)P_0^{-1} = P_0^m L_m$, the null spaces for L_m and $P_0 - P_0^m$ can be different. Thus, the error covariance of an unobservable state can experience a decrease depending on the choice of the initial error covariance matrix. Let the SVD of L_m be $U_m \Sigma_m U_m^T$. Decompose $U_m = \begin{bmatrix} U_{mo} & U_{mu} \end{bmatrix}$ and $\Sigma_m = \begin{bmatrix} \Sigma_{mo} & 0 \\ 0 & 0 \end{bmatrix}$ such that U_{mo} is the matrix formed by the observable singular vectors corresponding to the positive singular values in $\Sigma_{mo} > 0$, and U_{mu} is the matrix formed by the unobservable singular vectors corresponding to zero singular values in Σ_m. Let $P_{0,ou} = U_{mo}^T P_0 U_{mu}$ such that $P_{0,ou}$ is the cross-correlation of observable and unobservable states. Then $U_{mu}^T\left(P_0 - P_0^m\right)U_{mu} = P_{0,ou}^T M_{mo} P_{0,ou}$, $M_{mo} = \left(U_{mo}^T P_0 U_{mo} + \Sigma_{mo}^{-1}\right)^{-1}$. Thus, if there exist nonzero cross-correlation between observable and unobservable states in the initial error covariance matrix such that $U_{mo}^T P_0 u_{mu} \neq 0$ where u_{mu} is an unobservable singular vector, then, since $M_{mo} > 0$, $u_{mu}^T\left(P_0 - P_0^m\right)u_{mu} = u_{mu}^T P_0 U_{mo} M_{mo} U_{mo}^T P_0 u_{mu} > 0$.

Next, the size of change in error covariance matrix is presented with SVD. It can be shown that $P_0 - P_0^m = P_0 U_{mo} M_{mo} U_{mo}^T P_0 = P_0 U_m M_m U_m^T P_0$ with $M_m = \begin{bmatrix} M_{mo} & 0 \\ 0 & 0 \end{bmatrix}$. Then we have the following theorem:

Theorem 2: *The upper and lower bounds for* $P_0 - P_0^m$ *are*

$$\frac{\bar{\sigma}(P_0)\underline{\sigma}(P_0)\underline{\sigma}(\Sigma_m)}{1+\underline{\sigma}(\Sigma_m)\bar{\sigma}(P_0)} \le \bar{\sigma}\left(P_0 - P_0^m\right) \le \bar{\sigma}^2(P_0)\min\left\{\frac{1}{\underline{\sigma}(P_0)}, \bar{\sigma}(\Sigma_m)\right\}$$

$$\frac{\underline{\sigma}^2(P_0)\underline{\sigma}(\Sigma_m)}{1+\underline{\sigma}(\Sigma_m)\bar{\sigma}(P_0)} \le \underline{\sigma}\left(P_0 - P_0^m\right) \le \bar{\sigma}(P_0)\underline{\sigma}(P_0)\min\left\{\frac{1}{\underline{\sigma}(P_0)}, \bar{\sigma}(\Sigma_m)\right\}$$

The theorem shows that the bounds of the singular values of $P_0 - P_0^m$ can be greatly influenced by the choice of initial error covariance. In general, if the size of initial error covariance is small, its inverse can be large and the influence of information matrix on the error covariance can be small depending on its relative size with respect to that of the inverse of initial covariance matrix. Theorem 2 can also be useful to investigate the influence of model perturbations on the error covariance. Let P_0 be the error covariance matrix of unperturbed system and L_m be the perturbation of information matrix due to errors in the system model or computation. In this case L_m is a symmetric non-definite matrix. Thus, the non-negative matrix Σ_m is replaced by the non-definite diagonal matrix composed of eigenvalues of L_m, and Σ_{mo} is formed by the non-zero eigenvalues.

3. Application to GPS/INS

In this section the influence of model errors in GPS/INS on the singular structure of information matrix is introduced for a simple vehicle motion. For simplicity of notation, exponential numbers with 10 as base of the power are expressed with E such that 2.0E-05 means 2.0×10^{-5} in this section.

The relatively exact state space model for error in the INS mechanization in the Earth-centered earth-fixed (ECEF) frame is [11]

$$\dot{x} = Ax, \; y = Cx + v \tag{11}$$

with

$$x = \begin{bmatrix} \left(\delta P^e\right)^T & \left(\delta V^e\right)^T & \left(\gamma^b\right)^T & \varepsilon_g^T & \varepsilon_a^T & \delta l^T \end{bmatrix}^T \tag{12}$$

$$A = \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 \\ G^e & -2\Omega_{ie}^e & -R_b^e F^b & 0 & R_b^e & 0 \\ 0 & 0 & -\Omega_{ib}^b & I & 0 & 0 \\ 0 & 0 & 0 & -\sigma_g I & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sigma_a I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{13}$$

$$C = \begin{bmatrix} I & 0 & -R_b^e L^b & 0 & 0 & R_b^e \end{bmatrix}. \tag{14}$$

where $G^e = \partial g^e / \partial P^e$, δP^e and δV^e are the position and velocity errors, respectively, in the ECEF frame, γ^b is the attitude error in the body frame such that $\hat{R}_b^e = R_b^e\left(I + \left[\gamma^b \times\right]\right)$, ε_g and ε_a are bias errors for the gyro and accelerometer, respectively, δl is the error for the lever arm in the body frame between the inertial sensors and GPS antenna, F^b is the cross product matrix of the specific force f^b, σ_g and σ_a are the inverses of the correlation times of noises in the gyro and accelerometer, respectively, L^b is the cross product matrix of the lever arm in the body frame, and g^e is the gravity in the inertial frame. There are several simplified models for the system (A, C). For the comparison with very simple form in [11], the system model in (15) is considered in this paper. Comparison is made on the singular structure of information matrices for the systems (A, C) and (A_s, C) for a vehicle that moves at 100 m/s to the north during 17 seconds. Measurement sampling period is one second.

$$A_s = \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & -R_b^e F^b & 0 & R_b^e & 0 \\ 0 & 0 & -\Omega_{eb}^b & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (15)$$

The square roots of the singular values of information matrices are given in Table I. Singular vectors for (A,C) are given in Tabs. II-1 through II-3, and those for (A_s,C) are given in Tabs. III-1 through III-3. In the singular vector tables, elements whose absolute values are smaller than 0.1 are omitted. From the information matrix alone, error covariance can be calculated for the case in which there is no knowledge on the initial state. The standard deviations (STDs) of gyro bias error vector for this case are (1.08, 4.65, 6.93E+2) and (2.09E-4, 1.01, 1.13E+3) in rad/s for (A,C) and (A_s,C), respectively. The square roots of the measure of observability for the gyro bias error vector are (4.46E+5, 4.46E+5, 2.63E+4) and (4.63E+5, 4.63E+5, 2.71E+4) in s/rad for (A,C) and (A_s,C), respectively. Thus, while the error covariance of the gyro bias is very sensitive to the system model change, the observability measure for it is quite insensitive. In both models, STD of the vertical component of gyro bias error is so large that it can be considered unobservable in practice. It is shown in the tables that even though changes in the singular values are relatively small, those for singular vectors corresponding to the singular values 1,2,7,8, and 12-17 are large. However, the changes in the subspace spanned by the group of singular vectors corresponding to the singular values whose size are similar are small. Note that the sizes of singular values for each group of 1 and 2, 7 and 8, and 12-17 are similar. To express changes in subspaces, let the distance between two subspaces be defined by the largest principle angle between them [16]. The distances between the subspaces for (A,C) and (A_s,C) that are spanned by the three groups of singular vectors are 0, 0, and 4.4 degrees, respectively. The relatively large changes in the group of singular vectors corresponding to the small singular values can have negligible effect on the error covariance if the size of initial error covariance is small.

Table I: Square roots of the information matrix singular values

No	(A,C)	(A_s,C)
1	4.51E+5	4.68E+5
2	4.51E+5	4.68E+5
3	1.49E+4	1.53E+4
4	1.49E+4	1.53E+4
5	4.46E+3	4.82E+3
6	2.86E+2	2.88E+2
7	1.79E+2	1.84E+2
8	1.76E+2	1.80E+2
9	1.10E+2	1.10E+2
10	7.16E+1	7.23E+1
11	4.95E+1	5.01E+1
12	9.9E-1	3.18E-3
13	2.70E-1	7.76E-4
14	3.60E-3	6.10E-4
15	2.60E-3	5.94E-4
16	1.45E-3	1.46E-4
17	1.37E-4	4.58E-5
18	8.35E-6	5.15E-6

Table II-1: Singular vectors of the information matrix for (A,C)

Singular value No.					
1	2	3	4	5	6
					-0.23
					-0.16
					-0.38
					-0.27
	-0.19	-0.69	0.68		
-0.19		-0.68	-0.70		
					0.40
-0.35	0.92	0.15	-0.14		
-0.92	-0.35	0.15	0.14		
					0.67
				0.99	

Table II-2: Singular vectors of the information matrix for (A,C)

Singular value No.					
7	8	9	10	11	12
0.17		0.34	-0.15	0.32	
0.32	0.25		-0.40	-0.41	
-0.30		0.24	0.29	-0.40	
0.33	-0.38	-0.26	0.20	-0.18	-0.65
0.40	0.70		0.53	0.22	
-0.54	0.43	-0.18	-0.38	0.22	-0.45
		-0.58			
		0.47			-0.55
					0.26
	0.14				
-0.33		0.29	0.36		
-0.33		-0.29	0.37		
	-0.25			0.66	

Table II-3: Singular vectors of the information matrix for (A,C)

Singular value No.					
13	14	15	16	17	18
		-0.80	0.17		
				-0.70	
	0.12		0.75		
	-0.18				
	-0.13				
		-0.31	0.27		0.58
	-0.16				
0.26	-0.92				
0.96	0.25				
			-0.35	-0.42	0.57
		-0.30			-0.58
		0.42	0.43	0.43	

Table III-1: Singular vectors of the information matrix for (A_s,C)

Singular value No.					
1	2	3	4	5	6
					-0.23
					-0.16
					-0.39
					-0.27

-0.16	-0.11	-0.69	0.68		
-0.11	-0.16	-0.69	-0.69		
					0.40
-0.82	0.54	0.14	-0.14		
-0.54	-0.82	0.14	0.14		
					0.67
				0.99	
					-0.20
					0.20

Table III-2: Singular vectors of the information matrix for (A_s, C)

Singular value No.					
7	8	9	10	11	12
0.14	-0.12	0.34	-0.15	0.32	-0.19
-0.10	-0.38		-0.41	-0.42	
-0.18	0.25	0.23	0.29	-0.40	-0.76
0.48	-0.15	-0.27	0.20	-0.17	
-0.50	-0.65		0.52	0.21	
-0.61	0.33	-0.19	-0.38	0.21	
		-0.58			-0.28
		0.46			-0.55
					0.26
-0.14					
	0.33	0.29	0.37		0.37
	0.33	-0.29	0.37		
0.24				0.66	-0.38

Table III-3: Singular vectors of the information matrix for (A_s, C)

Singular value No.					
13	14	15	16	17	18
-0.63	-0.42	-0.23			
-0.42	0.53	0.18			
	-0.25				
	-0.18	0.42			
				-0.10	
-0.24	-0.14	-0.31			-0.58
	-0.22	0.53			
0.11			0.98		
				0.99	
-0.26	0.28				-0.58
-0.51	0.14			-0.10	0.57
	0.53	-0.23	0.13		

Figs. 1-4 show the standard deviations (STD) of the gyro bias estimation error for various conditions. Figs. 1-2 are for the accurate system model (A, C) and Figs. 3-4 are for the simplified system model (A_s, C). The initial error covariance matrix for Figs. 1 and 3 is a diagonal matrix with the following STDs: 1m for each element of position and lever arm error vectors, 1 cm/s for each element of velocity error vector, 1.0E-4 rad for roll and pitch errors, 0.5 deg for yaw error, 1.0E-5 rad/s for each element of gyro bias error vector, and 0.1mg for each element of accelerometer bias error vector. The initial error covariance matrix for Figs. 2 and 4 is the identity matrix. In the figures, STDs of the gyro bias error in the two models do not show large differences for the same initial covariance matrix. However, they are very sensitive to the initial error covariance change.

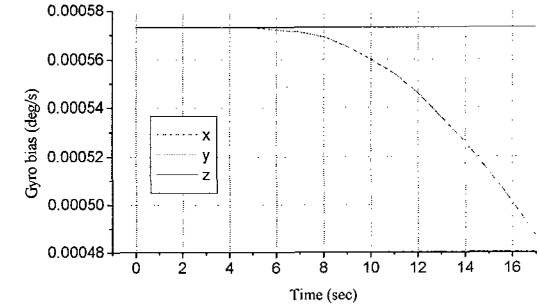

Fig. 1. STD of gyro bias estimation error for exact error dynamics model

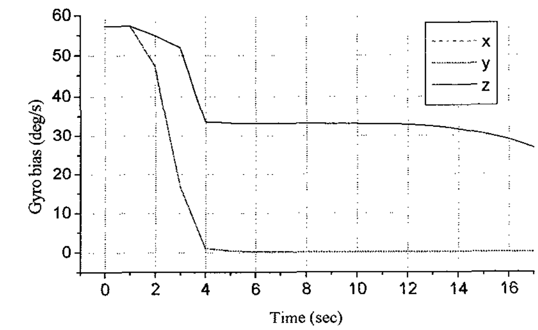

Fig. 2. STD of gyro bias estimation error for exact error dynamics model

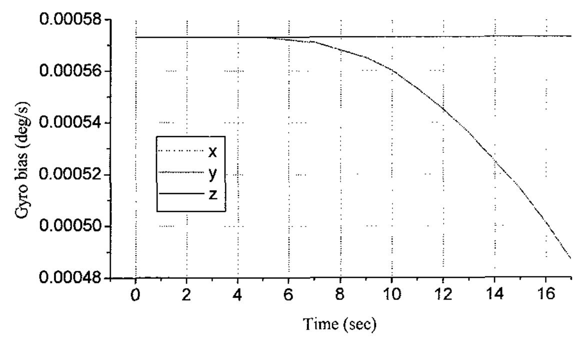

Fig. 3. STD of gyro bias estimation error for the simplified error dynamics model

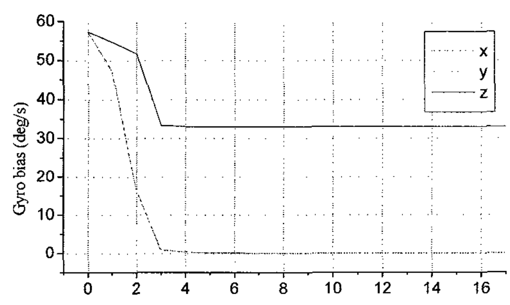

Fig. 4. STD of gyro bias estimation error for the simplified error dynamics model

4. Conclusion

In this paper a measure of observability that is less sensitive to perturbation is introduced. The relations among the measure of observability, information matrix, error covariance, and system model perturbation are given with singular value decomposition techniques. The behavior of error covariance can be reasonably expected from the observability measure and initial error covariance. The measure can be applied to general time-varying multi-input and multi-output systems.

A numerical example of the observability analysis of GPS/INS is given. It is confirmed that the measure of observability is less sensitive to the system model perturbation. Both the observability analysis and covariance simulation show that the vertical component of gyro bias can be considered unobservable for constant speed horizontal motion with small initial error covariance such as in the tactical or higher grade INS.

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