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Distances between Interval-valued Intuitionistic Fuzzy Sets

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Abstract

We give a geometrical interpretation of the interval-valued fuzzy set. So, based on the geometrical background, we propose new distance measures between interval-valued fuzzy sets and compare these measures with distance measures proposed by Burillo and Bustince and Grzegorzewski, respectively. Furthermore, we extend three methods for measuring distances between interval-valued fuzzy sets to interval-valued intuitionistic fuzzy sets.

1. Introduction

Most of problems in real life situation such as economics, engineering, environment, social sciences and medical sciences not always involve crisp data. So we cannot successfully use the traditional methods because of various types of uncertainties presented in those problems. Since Zadeh [17] introduced fuzzy sets in 1965, many approaches and theories treating imprecision and uncertainty have been proposed. Some of these theories, like as intuitionistic fuzzy set theory and interval-valued fuzzy set theory and interval-valued intuitionistic fuzzy set theory, are extensions of fuzzy set theory and the others try to handle imprecision and uncertainty in different ways. Some authors [3,6] pointed out that there is strong connection between intuitionistic fuzzy sets and interval-valued fuzzy sets, i.e., intuitionistic fuzzy set theory and interval-valued fuzzy set theory are equipollent generalizations of fuzzy set theory.

Some authors have investigated interval-valued fuzzy set and its relevant topics, for example, Burillo and Bustince [4] researched entropy and distance for interval-valued fuzzy sets, Grzegorzewski [7] studied distance between interval-valued fuzzy sets based on the Hausdorff metric, Zeng and Li [20] studied the relationship between entropy and similarity measure of interval-valued fuzzy sets. In this paper, we give a geometrical interpretation of the interval-valued fuzzy set and take into account all three parameters describing the interval-valued fuzzy set. So, based on the geometrical background, we propose new distance measures between interval-valued fuzzy sets and compare these measures with above-mentioned distance measures proposed by Burillo and Bustince [4] and Grzegorzewski [7], respectively. Furthermore, we extend three methods for measuring distances between interval-valued fuzzy sets to interval-valued intuitionistic fuzzy sets.

2. Distances between Interval-valued intuitionistic fuzzy sets

As a generalization of the notion of intuitionistic fuzzy sets, Atanassov and Gargov [3] introduced the notion of interval-valued intuitionistic fuzzy sets in the spirit of interval-valued fuzzy sets.

An interval-valued intuitionistic fuzzy set (**IVIF** set, for short) A on a universe X is an object having the form

$A = \{(x, M_A(x), N_A(x)) : x \in X\}$,

where $M_A : X \to [I]$ and $N_A : X \to [I]$ denote, respectively, membership function and non-membership function of A and satisfy $0 \le M_{AU}(x) + N_{AU}(X) \le 1$ for any $x \in X$.

Let **IVIF**(X) denote all interval-valued fuzzy sets on X. Even though we can represent a fuzzy set in an intuitionistic-type representation, we can not always represent any interval-valued fuzzy set in interval-valued intuitionistic-type representation. For example, let A be an inter-valued fuzzy set in $X=\{x\}$ such that $M_A=[\frac{1}{4},\frac{1}{2}]$. Then $(M_A,\overline{M}_A)=([\frac{1}{4},\frac{1}{2}],\ [\frac{1}{2},\frac{3}{4}])$ is not an **IVIF** set because $M_{AU}+\overline{M}_{AU}=\frac{1}{2}+\frac{3}{4}\not\leq 1$. However, if an IVF set A satisfy the condition $M_{AU}+\overline{M}_{AU}\leq 1$ i.e. $M_{AU}+1-M_{AL}\leq 1$, then the interval-valued fuzzy set A can represent interval-valued intuitionistic-type representation $(M_A,\overline{M}_A)$.

We extend the Burillo and Bustince's distances to **IVIF** sets. For any two **IVIF** sets $A=\{(x_i,M_A(x_i),N_A(x_i)):\ x_i\in X\}$ and $B=\{(x_i,M_B(x_i),N_B(x_i)):\ x_i\in X\}$ of the universe of discourse $X=\{x_1,x_2,\cdots,x_n\}$,

- the Hamming distance $d_1'(A,B)$:

$$d_1'(A,B)=\frac{1}{4}\sum_{i=1}^{n}[|M_{AL}(x_i)-M_{BL}(x_i)|+|M_{AU}(x_i)-M_{BU}(x_i)|+|N_{AL}(x_i)-N_{BL}(x_i)|+|N_{AU}(x_i)-N_{BU}(x_i)|], \quad (2)$$

- the normalized Hamming distance $l_1'(A,B)$:

$$l_1'(A,B)=\frac{1}{4n}\sum_{i=1}^{n}[|M_{AL}(x_i)-M_{BL}(x_i)|+|M_{AU}(x_i)-M_{BU}(x_i)|+|N_{AL}(x_i)-N_{BL}(x_i)|+|N_{AU}(x_i)-N_{BU}(x_i)|], \quad (3)$$

- the Euclidean distance $e_1'(A,B)$:

$$e_1'(A,B)=\{\frac{1}{4}\sum_{i=1}^{n}[(M_{AL}(x_i)-M_{BL}(x_i))^2+(M_{AU}(x_i)-M_{BU}(x_i))^2+(N_{AL}(x_i)-N_{BL}(x_i))^2+(N_{AU}(x_i)-N_{BU}(x_i))^2]\}^{\frac{1}{2}}, \quad (4)$$

- the normalized Euclidean distance $q_1'(A,B)$:

$$q_1'(A,B)=\{\frac{1}{4n}\sum_{i=1}^{n}[(M_{AL}(x_i)-M_{BL}(x_i))^2+(M_{AU}(x_i)-M_{BU}(x_i))^2+(N_{AL}(x_i)-N_{BL}(x_i))^2+(N_{AU}(x_i)-N_{BU}(x_i))^2]\}^{\frac{1}{2}}, \quad (5)$$

Now, we consider the amplitude margin to modify these distances.

- the Hamming distance $d_1''(A,B)$:

$$d_1''(A,B)=\frac{1}{4}\sum_{i=1}^{n}[|M_{AL}(x_i)-M_{BL}(x_i)|+|M_{AU}(x_i)-M_{BU}(x_i)|+|N_{AL}(x_i)-N_{BL}(x_i)|+|N_{AU}(x_i)-N_{BU}(x_i)|+|W_{M_A}(x_i)-W_{M_B}(x_i)|+|W_{N_A}(x_i)-W_{N_B}(x_i)|], \quad (6)$$

- the normalized Hamming distance $l_1'(A,B)$:

$$l_1''(A,B)=\frac{1}{4n}\sum_{i=1}^{n}[|M_{AL}(x_i)-M_{BL}(x_i)|+|M_{AU}(x_i)-M_{BU}(x_i)|+|N_{AL}(x_i)-N_{BL}(x_i)|+|N_{AU}(x_i)-N_{BU}(x_i)|+|W_{M_A}(x_i)-W_{M_B}(x_i)|+|W_{N_A}(x_i)-W_{N_B}(x_i)|], \quad (7)$$

- the Euclidean distance $e_1''(A,B)$:

$$e_1''(A,B)=\{\frac{1}{4}\sum_{i=1}^{n}[(M_{AL}(x_i)-M_{BL}(x_i))^2+(M_{AU}(x_i)-M_{BU}(x_i))^2+(N_{AL}(x_i)-N_{BL}(x_i))^2+(N_{AU}(x_i)-N_{BU}(x_i))^2+(W_{M_A}(x_i)-W_{M_B}(x_i))^2+(W_{N_A}(x_i)-W_{N_B}(x_i))^2\}^{\frac{1}{2}}. \quad (8)$$

- the normalized Euclidean distance $q_1''(A,B)$:

$$q_1''(A,B)=\{\frac{1}{4n}\sum_{i=1}^{n}[(M_{AL}(x_i)-M_{BL}(x_i))^2+(M_{AU}(x_i)-M_{BU}(x_i))^2+(N_{AL}(x_i)-N_{BL}(x_i))^2+(N_{AU}(x_i)-N_{BU}(x_i))^2+(W_{M_A}(x_i)-W_{M_B}(x_i))^2+(W_{N_A}(x_i)-W_{N_B}(x_i))^2. \quad (9)$$

Clearly these distances satisfy the conditions of the metric (cf. [8]). Finally, we extend the Grzegorzewski's distances to **IVIF** sets as (?)-(?). For any two **IVIF** sets $A=\{(x_i,M_A(x_i),N_A(x_i)):\ x_i\in X\}$ and $B=\{(x_i,M_B(x_i),N_B(x_i)):\ x_i\in X\}$ of the universe of discourse $X=\{x_1,\ x_2,\cdots,x_n\}$,

- the Hamming distance $d_H(A,B)$:

$$d_H(A,B)=\frac{1}{2}\sum_{i=1}^{n}[\max\{|M_{AL}(x_i)-M_{BL}(x_i)|,\ |M_{AU}(x_i)-M_{BU}(x_i)|\}+\max\{|N_{AL}(x_i)-N_{BL}(x_i)|,\ |N_{AU}(x_i)-N_{BU}(x_i)|\}], \quad (10)$$

- the normalized Hamming distance $l_H(A,B)$:

$$l_H(A,B)=\frac{1}{2n}\sum_{i=1}^{n}[\max\{|M_{AL}(x_i)-M_{BL}(x_i)|,\ |M_{AU}(x_i)-M_{BU}(x_i)|\}+\max\{|N_{AU}(x_i)-N_{BU}(x_i)|.$$

$$|N_{AU}(x_i) - N_{BU}(x_i)|\}], \quad (11)$$

- the Euclidean distance $e_H(A,B)$:

$$e_H(A,B) = \{\frac{1}{2}\sum_{i=1}^{n}[(\max\{|M_{AL}(x_i) - M_{BL}(x_i)|, |M_{AU}(x_i) - M_{BU}(x_i)|\})^2 + (\max\{|N_{AL}(x_i) - N_{BL}(x_i)|, |N_{AU}(x_i) - N_{BU}(x_i)|\})^2]\}^{\frac{1}{2}}, \quad (12)$$

- the normalized Euclidean distance $q_H(A,B)$:

$$q_H(A,B) = \{\frac{1}{2}\sum_{i=1}^{n}[(\max\{|M_{AL}(x_i) - M_{BL}(x_i)|, |M_{AU}(x_i) - M_{BU}(x_i)|\})^2 + (\max\{|N_{AL}(x_i) - N_{BL}(x_i)|, |N_{AU}(x_i) - N_{BU}(x_i)|\})^2]\}^{\frac{1}{2}}. \quad (13)$$

Proposition 1. Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite universe of discourse. Then function d_H, l_H, e_H, q_H : **IVIF**$(X) \rightarrow \mathbb{R}^+ \cup \{0\}$ given by (10)-(13), respectively, are metrics.

Proposition 2. For any two **IVIF** sets $A = \{(x_i, M_A(x_i), N_A(x_i)) : x_i \in X\}$ and $B = \{(x_i, M_B(x_i), N_B(x_i)) : x_i \in X\}$ of the universe of discourse $X = \{x_1, x_2, \cdots, x_n\}$, the following inequalities hold:

$$d_H(A,B) \le n, \quad (14)$$
$$l_H(A,B) \le 1, \quad (15)$$
$$e_H(A,B) \le \sqrt{n}, \quad (16)$$
$$q_H(A,B) \le 1. \quad (17)$$

Proposition 3. For any two **IVIF** sets $A = \{(x_i, M_A(x_i), N_A(x_i)) : x_i \in X\}$ and $B = \{(x_i, M_B(x_i), N_B(x_i)) : x_i \in X\}$ of the universe of discourse $X = \{x_1, x_2, \cdots, x_n\}$, the following inequalities hold:

$$d_1'(A,B) \le d_H(A,B) \le d_1''(A,B), \quad (18)$$
$$l_1'(A,B) \le l_H(A,B) \le l_1''(A,B), \quad (19)$$
$$e_1'(A,B) \le e_H(A,B) \le e_1''(A,B), \quad (20)$$
$$q_1'(A,B) \le q_H(A,B) \le q_1''(A,B). \quad (21)$$

When generalizing any notion it is desirable that the new object should be consistent with the primary one and it should reduce to that primary one in some particular cases. As it was mentioned above each **IVF** set can be **IVIF** set under some conditions.
Thus it would be desirable that our definitions (2)-(13) should reduce to the Burillo and Bustince's distances, our distances and Grzegorzewski's distances, respectively, for ordinary **IVF** sets. One can check easily that

Proposition 4. For any two **IVIF** sets $A, B \in X = \{x_1, x_2, \cdots, x_n\}$ such that $A = \{(x_i, M_A(x_i), N_A(x_i)) : x_i \in X\}$ and $B = \{(x_i, M_B(x_i), N_B(x_i)) : x_i \in X\}$, the following equalities hold:

$$d'(A,B) = d_1'(A,B), \quad (22)$$
$$l'(A,B) = l_1'(A,B), \quad (23)$$
$$e'(A,B) = e_1'(A,B), \quad (24)$$
$$q'(A,B) = q_1'(A,B), \quad (25)$$
$$d''(A,B) = d_1''(A,B), \quad (26)$$
$$l''(A,B) = l_1''(A,B), \quad (27)$$
$$e''(A,B) = e_1''(A,B), \quad (28)$$
$$q''(A,B) = q_1''(A,B), \quad (29)$$
$$d_h(A,B) = d_H(A,B), \quad (30)$$
$$l_h(A,B) = l_H(A,B), \quad (31)$$
$$e_h(A,B) = e_H(A,B), \quad (32)$$
$$q_h(A,B) = q_H(A,B). \quad (33)$$

Remark 1. Since intuitionistic fuzzy sets and **IVF** sets are equipollent generalizations of fuzzy sets, our definitions (2)-(13) should also reduce to the Szmidt and Kacprzyk's distances [11] and Grzegorzewski 's distances [7], respectively, for ordinary intuitionistic fuzzy sets.

3. Conclusions

In this paper, we propose new distances between **IVF** sets by taking into account three parameters describing an **IVF** set. We compare these distances with distances proposed by Burillo and Bustince and Grzegorzewski, respectively. Furthermore, we extend three methods for measuring distances between **IVF** sets to **IVIF** sets and show that these reduce to the Burillo and Bustince's distances, our distances and Grzegorzewski's distances, respectively, for ordinary **IVF** sets.

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