# 구간치 쇼케이적분과 위험률 가격 측정에서의 응용

# Interval-valued Choquet integrals and applications in pricing risks

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**Abstract**

Non-additive measures and their corresponding Choquet integrals are very useful tools which are used in both insurance and financial markets. In both markets, it is important to to update prices to account for additional information. The update price is represented by the Choquet integral with respect to the conditioned non-additive measure. In this paper, we consider a price functional H on interval-valued risks defined by interval-valued Choquet integral with respect to a non-additive measure. In particular, we prove that if an interval-valued pricing functional H satisfies the properties of monotonicity, comonotonic additivity, and continuity, then there exists an two non-additive measures $\mu_1, \mu_2$ such that it is represented by interval-valued choquet integral on interval-valued risks.

**Key words** : Choquet integrals, non-additive measures, interval-valued risks, updated prices.

## 1. Introduction

Murofushi and Sugeno [9] have studied some characterizations of Choquet integrals. Choquet integral allow to define Choquet price, Tolerant or intorant character, and insurance price, etc.(see[8,10,11,12,13,14]).

The set-valued Choquet integral was first introduced by Jang, Kil, Kim and Kwon [3] and restudied by Zhang, Guo and Lia [15]. Theory about set-valued integrals has drawn much attention due to numerous applications in mathematical economics, information theory, expected utility theory, expected utility theory, and risk analysis(see[1,2,15]). Based on this, we have been researching interval-valued integrals and giving various formulas used in the above area mentioned (see[3,4,5,6,7]).

We note that non-additive measures and their corresponding Choquet integrals are very useful tools which are used in both insurance and financial markets. In both markets, it is important to update prices to account for additional information. The update price is represented by the Choquet integral with respect to a non-additive measure.

In this paper, we define igned interval-valued Choquet intgerals with respected to a non-additive measure and investigate some applications in pricing risks. We also introduce a price functional $H$ defined by Choquet inetgral on interval-valued risks. We note that interval-valued risks mean ambiguity risks.

In section 2, we list various definitions and notations of interval numbers, interval-valued risks, signed Choquet integrals, and signed interval-valued Choquet integrals. In section 3, we introduce the concept of the symmetric Choquet integral and define the symmetric interval-valued Choquet integral. We also define a price functional $H$ defined by Choquet integral on interval-valued risks and discuss some characterizations of them. Furthermore, we prove that that if an interval-valued pricing functional H satisfies the properties of monotonicity, comonotonic additivity, and continuity, then there exists an two non-additive measures $\mu_1, \mu_2$ such that it is represented by interval-valued choquet integral on interval-valued risks.

## 2. Preliminaries and Definitions

Let $\Omega$ be the space of outcomes, $\Psi$ the set of all risks, and $\Im$ a $\sigma$-algebra of subsets of $\Omega$.

**Definition 2.1.** (1) A set function $\mu$on a measurable space $(\Omega,\Im)$ is called a non-additive measure if $\mu(\varnothing)=0$ and $\mu$ satisfies monotonicity(with respect to set inclusion), that is,

$$\mu(A) \le \mu(B),$$

whenever $A,B \in \Im$, $A \subset B$.

(2) μ is said to be lower semi-continuous if for every increasing sequence $\{A_n\}$ of measurable sets,

$$\mu(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n).$$

(3) μ is said to be upper semi-continuous if for every decreasing sequence $\{A_n\}$ of measurable sets and $\mu(A_1) < \infty$,

$$\mu(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n).$$

(4) μ is said to be continuous if it is both lower semi-continuous and upper semi-continuous.

(5) μ is said to be submodular if

$$\mu(A \cup B) + \mu(A \cap B) \le \mu(A) + \mu(B),$$

for all $A, B \in \Im$.

(6) $\mu$ is said to be subadditive if the above inequality holds for all $A, B \in \Im$ with $A \cap B = \varnothing$.

(7) If the reverse inequality holds, then $\mu$ is said to be supermodular (superadditive).

Because $\mu$ is defined on a $\sigma$-algebra, it has a conjugate, or dual, measure $\bar{\mu}$ on $\Im$, defined by

$$\bar{\mu}(A) = \mu(\Omega) - \mu(A^c),$$

in which $A^c = \Omega - A$.

**Definition 2.2** Let $x$ be a measurable real-valued function on $\Omega$ and $\mu$ a non-additive measure. Then we define the decumulative distribution function $G_{\mu,x} : \mathbb{R} \to [0,\infty)$ by

$$G_{\mu,x}(x) = \mu\{x > \alpha\},$$

for all $\alpha \in \mathbb{R}$.

We note that if $x$ represents monetary lossor gain, then call $x$ a risk; risks are not necessarily bounded(see [14]). When the functions $\mu$ and $x$ are clear from the context, we write $G$ for $G_{\mu,x}$.

**Definition 2.3.**([9,10,13]) (1) The Choquet integral of a risk $x$ is

$$C_\mu(x) = \int_{-\infty}^{0} [G(x) - \mu(\Omega)]d\alpha + \int_{0}^{\infty} G(\alpha)d\alpha,$$

in which the right-hand side is the sum of two(possibly improper) Riemann integrals.

(2) A risk $x$ is called c-integrable if the Choquet integral of $x$ can be defined and its value is finite.

**Definition 2.4.** Risks $x,y$ are said to be comonotonic, denoted by $x \sim y$ if there are no points $w,w' \in \Omega$ such that $x(w) > x(w')$ and $y(w) > y(w')$.

**Theorem 2.5.**([8,9,10,11,12,13,14]) Let $\mu$ be a non-additive measure and nonnegative risks $x,y : \Omega \to [0,\infty)$. Then we have as followings.

(1) If $x \le y$, then $C_\mu(x) \le C_\mu(y)$.

(2) If $x \sim y$ and $a,b \in [0,\infty)$, then

$$C_\mu(ax+by) = aC_\mu(x) + bC_\mu(y).$$

**Theorem 2.6.**([12,14]) Let $\bar{\mu}$ be a conjugate measure of a non-additive measure $\mu$.

(1) If $A \in \mathfrak{I}$ and $x$ is a risk, then $C_\mu(1_A) = \mu(A)$ and $C_\mu(-A) = -C_{\bar{\mu}}(x)$.
(2) If $x$ is a risk and $a \geq 0, b \in \mathbb{R}$, then
$$C_\mu(ax+b) = aC_\mu(x) + b\mu(\Omega).$$
(3) If $x,y$ are risks and $x \leq y$, then
$$C_\mu(x) \leq C_\mu(y).$$
(4) If $\mu$ is submodular, and if $x,y$ are bounded from below, then
$$C_\mu(x+y) \leq C_\mu(x) + C_\mu(y).$$
(5) If $x,y$ are comonotonic risks, then
$$C_\mu(x+y) \leq C_\mu(x) + C_\mu(y).$$

**Definition 2.7.** ([13]) Let $h: \Psi \to \overline{\mathbb{R}}$ be a pricing functional.
(1) $h$ is said to be monotone if $x,y \in \Psi$ and $x \leq y$, then $h(x) \leq h(y)$.
(2) $h$ is said to be comonotonic additive if $x,y \in \Psi$ and $x \sim y$, then
$$h(x+y) = h(x) + h(y).$$
(3) $h$ is said to be continuous if $x \in \Psi$ and $a \geq 0$; then
$$\lim_{\alpha \to 0^+} h(max(x-\alpha, 0)) = h(max(x,0)),$$
$$\lim_{\alpha \to \infty} h(min(x,\alpha)) = h(x), \text{ and}$$
$$\lim_{\alpha \to -\infty} h(max(x,\alpha)) = h(x).$$

**Theorem 2.8.**([13]) If a pricing functional $h$ on $\Psi$ satisfies the properties of monotonicity, comonotonic additivity, and continuity, then there is a non-additive measure $\mu$ on $(\Omega, \mathfrak{I})$ such that $h(x) = C_\mu(x)$ for all risks $x \in \Psi$.

## 3. Pricing functionals on interval-valued risks.

Throughout this paper, we denote
$$I(\mathbb{R}) = \{[a,b] \mid a,b \in \mathbb{R} \text{ and } a \leq b\}.$$
Then an element in $I(\mathbb{R})$ is called an interval number. On the interval numbers $I(\mathbb{R})$, we define the following operations; for each pair $[a,b],[c,d] \in I(\mathbb{R})$ and $k \in \mathbb{R}$,
$$[a,b] + [c,d] = [a+c, b+d],$$
$$[a,b] \cdot [c,d] = [a \cdot c \wedge a \cdot d \wedge b \cdot c \wedge b \cdot d, \; a \cdot c \vee a \cdot d \vee b \cdot c \vee b \cdot d],$$
$$k[a,b] = \begin{cases} [ka, kb], & k \geq 0 \\ [kb, ka], & k < 0 \end{cases},$$
$[a,b] \leq [c,d]$ if and only if $a \leq c$ and $b \leq d$,
$$\max\{[a,b],[c,d]\} \leq [a \vee c, b \vee d],$$
$$\min\{[a,b],[c,d]\} \leq [a \wedge c, b \wedge d].$$

It is easily to see that $(I(\mathbb{R}), d_H)$ is a metric space, where $d_H$ is the Hausdorff metric defined by
$$d_H(A,B) = \max\{sup_{a \in A} inf_{b \in B} |a-b|, \; sup_{b \in B} inf_{a \in A} |a-b|\}.$$
for all $A, B \in I(\mathbb{R})$ (see[6,7]). We denote $\bar{\mathfrak{I}}$ for the set of interval-valued risks from $\Omega$ to $I(\mathbb{R}) \backslash \{\varnothing\}$.

**Definition 3.1.** Let $X = [x_1, x_2]$ and $Y = [y_1, y_2] \in \bar{\mathfrak{I}}$ are said to be comonotonic, denoted by $X \sim Y$, if and only if $x_1 \sim y_1$ and $x_2 \sim y_2$.

**Definition 3.2.** The Choquet integral of an interval-valued risk $X \in \bar{\mathfrak{I}}$ with respect to a non-additive measure $\mu$ is defined by
$$C_\mu(X) = \{C_\mu(x) \mid x \in S(X)\}$$
where $S(X)$ is the set of $\mu$-a.e. measurable selections of $X$.
(2) $X$ is said to be c-integrable if
$$C_\mu(X) \neq \varnothing.$$
(3) $X$ is said to be c-integrably bounded if there is a c-integrable risk $x_0 \in \mathfrak{I}$ such that
$$\| X(w) \| = sup_{r \in X(w)} |r| \leq x_0(w)$$
for all $w \in \Omega$.

**Theorem 3.3** (Theorem 3.16 (iii) [15]) Let $\mu$ be a continuous non-additive measure. If a nonnegative interval-valued risk
$$X = [x_1, x_2] : \Omega \to I(\mathbb{R}^+) \backslash \{\varnothing\}$$
is c-integrably bounded, then $C_\mu(X)$ is an interval number, that is,
$$C_\mu(X) = [C_\mu(x_1), C_\mu(x_2)].$$

**Definition 3.4.** Let $\bar{\mu}$ be a conjugate measure of a non-additive measure $\mu$ and $H: \bar{\Psi} \to I(\overline{\mathbb{R}}) \backslash \{\varnothing\}$ an interval-valued pricing functional.

(1) $H$ is said to be monotone if $X, Y \in \bar{\Psi}$ and $X \leq Y$, then $H(X) \leq H(Y)$.

(2) $H$ is said to be comonotonic additive if $X, Y \in \bar{\Psi}$ and $X \sim Y$, then

$$H(X+Y) = H(X) + H(Y).$$

(3) $H$ is said to be continuous if $X \in \bar{\Psi}$ and $a \geq 0$; then

$$d_H - \lim_{\alpha \to 0^+} H[\max(x_1 - \alpha, 0), \max(x_2 - \alpha, 0)]$$

$$= H[\max(x_1, 0), \max(x_2, 0)]$$

$$d_H - \lim_{\alpha \to \infty} H[\min(x_1, \alpha), \min(x_2, \alpha)] = H(X),$$

and

$$d_H - \lim_{\alpha \to -\infty} H[\max(x_1, \alpha), \max(x_2, \alpha)] = H(X).$$

**Theorem 3.5.** If we define an interval-valued pricing functional $H^c : \bar{\Psi} \to I(\bar{\mathbb{R}}) \backslash \{\varnothing\}$ by

$$H^c(X) = [h_1^c(x_1), h_2^c(x_2)]$$

for all $X = [x_1, x_2] \in \bar{\Psi}$, then $H^c$ is monotone, comonotonic additive, and continuous on $\bar{\Psi}$.

**Theorem 3.6.** Let $H : \bar{\Psi} \to I(\bar{\mathbb{R}}) \backslash \{\varnothing\}$ be an interval-valued pricing functional. If $H$ satisfies the properties of monotonicity, comonotonic additivity, and continuity, then there exist two non-additive measures $\mu_1$ and $\mu_2$ such that

$$H(X) = [C_{\mu_1}(x_1), C_{\mu_2}(x_2)]$$

for all $X = [x_1, x_2] \in \bar{\Psi}$.

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