A Study on Theoretical aspects of Anisotropic meshing

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이방성 메쉬에 대한 이론적 연구

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Abstract

It is not always convenient to consider isotropic meshes where the edge length depends on the orientation of the edge. It is desirable to go for anisotropic mesh strategy instead. There are many instances where the solution shows directional features such as great variations along certain directions with less significant changes along other ones. Anisotropic meshes considered to be worthy in these cases. By using anisotropic elements we can resolve the solution accurately with few elements. Many techniques have been used as a common feature that the shape, size and orientation of the triangle elements are controlled by specifying metric tensor. This paper attempts at clear understanding of the estimation and equidistribution of the error and discusses the parameters like reliability and effectivity of an anisotropic mesh in a mathematical manner. Also we study some of applications of anisotropic mesh.

1. Introduction

Many physical problems exhibit anisotropic solution features which changes from one direction to another[4]. A properly chosen anisotropic mesh can be advantageous for significant gain on improvements in accuracy and efficiency. It is desirable to use anisotropic elements, where the edge length depends on the orientation of the edge[8]. By using anisotropic elements we can resolve the solution accurately with few number of elements. Distribution of anisotropic elements in a mesh is controlled by specifying a symmetric metric tensor. It is this tensor which controls shape, size and orientation of the elements. In general these metrics are constructed upon error estimation.

Generalizing anisotropic mesh adapting strategies into four steps:

1. Calculating an approximate (numerical) solution.
2. estimating the error locally and globally.
3. Determining an appropriate aspect ratio and stretching direction of the finite elements.
4. Generating an improved mesh.

There are many instances where the solution shows directional features such as great variations along certain directions with less significant changes along other ones. Examples include those having boundary and inner layers, shock waves, contact interfaces, and edge singularities.

From the classical finite element theory, aspect ratio can be defined as a ratio of diameter of the finite element **k** and supremum of the diameters contained in k (s). (The aspect-ratio of a mesh is the maximal aspect-ratio among its elements). Anisotropic elements are characterized by [10]

$$\frac{de}{s} \to \infty \qquad (1)$$

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Where the limit can be considered as h (size function) $\rightarrow 0$ (near edges).
This paper attempts at clear understanding of the theoretical aspects of the anisotropic meshing.

2. Error estimation and equidistribution

Anisotropic metric is constructed using an *posteriori* error estimate based on discrete approximation of the Hessian of the solution at the mesh nodes. The approximation error between an exact solution and a computed finite element solution is difficult to estimate in general but, according to Cea's lemma, it is bounded by the interpolation error for elliptic problems [2]. Practically, this relation holds for a large class of problems and the interpolation error is commonly used as an error estimator for adaptive mesh generation.

For a langrange finite element discretization of the variable u, the interpolation function is defined as

$$\prod_h u(x) = \sum_i X_i(x) u(x_i) \qquad (2)$$

Where x_i is the location of the node i.
On expanding the variable u into a Taylor series at x

$$u(y) = u(x) + \Delta u(x).(y-x) + R_1(x,y) \qquad (3)$$

Remainder,

$$R_1(y,x) = \frac{1}{2}(y-x).H(u)(y-x) \qquad (4)$$

H is the Hessian (matrix of second derivatives), evaluated somewhere between points x and y.

$$(H) = \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y \partial x} & \frac{\partial^2 u}{\partial y^2} \end{pmatrix} \qquad (5)$$

Upon plugging x_i for y and expanding and using these equalities

$$\sum_i X_i(x) = 1, \qquad \sum_i X_i(x)(x_i) = x.$$

One obtains,

$$u(x) - \prod_h u(x) = -\sum_i X_i(x) R_1(x_i, x) \qquad (7)$$

(7)
And we compute,

$$\left| u(x) - \prod_h u(x) \right| \le C \max \left| R_1(x_i, x) \right| \qquad (8)$$

Where C is the generic constant.

Now again,

$$\left| u(x) - \prod_h u(x) \right| \le C \max\max \langle \vec{p}, \left| H(u) \right| \vec{p}^T \rangle \qquad (9)$$

$$\left| H(u) \right| = R \left| \Lambda \right| R^T = (e_1 e_2) \begin{pmatrix} |\lambda_1| & 0 \\ 0 & |\lambda_2| \end{pmatrix} (e_1 e_2)^T, \qquad (10)$$

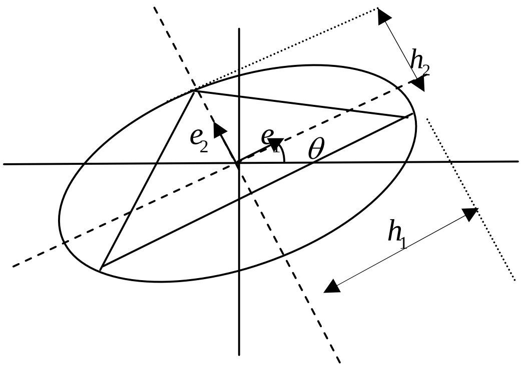

Fig. 1. Geometric interpretation of metric tensor

Components of the column matrix R are the eigenvectors, provide the directions of the semi-axis of the ellipse circumscribed by the element ***k***. Eigen values h_1 and h_2 measure the length of the semi-axes. Hence it can be inferred that shape and orientation of the evrey triangle can be controlled by eigenvectors and eigen values.

Now let M be a positive definite symmetric matrix such that,

$$\max \langle \vec{p}, \left| H(u) \right| \vec{p}^T \rangle \le \langle \vec{p}, M \vec{p}^T \rangle$$

Now (9) becomes ,

$$\left| u(x) - \prod_h u(x) \right| \le C \max \langle \vec{p}, M \vec{p}^T \rangle$$

This can be written as

$$\left| u(x) - \prod_h u(x) \right| \le C \left\| p^2 \right\|_M \qquad (11)$$

It follows that the errors of the interpolation will get bigger the higher the curvature of the function of the variable u and the bigger the elements. The above relation relates the interpolation error to the square of the largest edge length in k with respect to the metric M.

Now for equidistribution of error let [3],

$$\langle \vec{p}, M\vec{p}^T \rangle = \eta_e$$

$$\langle \vec{p}, \bar{M}\vec{p}^T \rangle = 1 \text{; where } \bar{M} = \frac{M}{\eta_e} = R\tilde{\Lambda}R^T;$$

$$\tilde{\Lambda} = diag\left(\tilde{\lambda}_i\right);$$

And $\tilde{\lambda}_i = \min\left(\max\left(\frac{C|\lambda_i|}{\eta_e}, h_{\max}^{-2} \right), h_{\min}^{-2} \right)$; and the target size along e_i is $h_i = \lambda_i^{-1/2}$. h_{max} and h_{min} are the maximum and minimum allowable target sizes while the constant C controls the error, and consequently final number of meshes. In (Yamakawa and Shimada 2000)anisotropy is defined by principal directions and an aspect ratio in each direction. The principal directions are represented by $\vec{e}_1$ and $\vec{e}_2$ as shown in fig.1, and in these directions the amounts of stretching of a mesh element are represented by two scalar values $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ [1].

Metric adaptation algorithms perform local or global operations to enforce the target size, stretching, and orientation prescribed by the control metric.

3. Edge length

For an edge p_1p_2 in the domain let $\vec{p}_t = (1-t)\, p_1 + tp_2$ be a parametric description for the segment p_1p_2, then its metric length defined as

$$\int_0^1 \sqrt{\left(p_2 - p_1\right)^T M\left(\vec{p}_t\right)\left(p_2 - p_1\right)}dt\,.$$

It has been shown that the adaptation process is equivalent to requiring all the mesh edges to have a unit metric length [2]. This is the reason for perfectly adaptive meshes being called as *unit meshes.*

By linearly interpolating metric $M\left(\vec{p}_t\right) = (1-t)\, M\left(p_1\right) + tM\left(p_2\right)$, approximating segment length as,

$$\left\|p_2 - p_1\right\|_M = \int_0^1 \sqrt{l_1^2 + t\left(l_2^2 - l_1^2\right)}dt = \frac{2}{3}\frac{l_1^2 + l_1l_2 + l_2^2}{l_1 + l_2} \tag{12}$$

Where $l_{i=1,2} = \sqrt{\left(p_2 - p_1\right)^T M\left(\vec{p}_i\right)\left(p_2 - p_1\right)}$

4. Reliability and effectivity

For any mesh on domain, it is desirable to define two important terms namely: reliability and effectivity[10]. As stated earlier the error first estimated element wise $\left(\eta_e\right)$ which then leads to the estimation of global error $\left(\eta\right)$.

We designate true error per element $\left|u(x) - \prod_h u(x)\right|$ as tolerance λ_e, then

The error estimator should not undermine the λ_e in the norm of a space of element. This is reliability for an element.

That is, $\eta_e \geq \lambda_e$. (13)

Anisotropic Error estimator for an element can be given by[9]

$$\eta_e = 2\,area_k^{1/p}\left|\lambda_1\left(p_0\right)\right| h_1^2 \tag{14}$$

Then substituting for area of the element we can have following equation

$$\eta_e = 2\left(\frac{1}{2}\sqrt{\left|\bar{\bar{M}}\right|}\left(p_1 - p_2\right)\times\left(p_2 - p_3\right)\right)_k^{1/p}\left|\lambda_1\left(p_0\right)\right| h_1^2 \tag{15}$$

The global error estimator is given by,

$$\eta = \left(\sum_k \eta_e^p\right)^{1/p} \tag{16}$$

For Euclidean norm ,

$$\eta = \left(\sum_k \eta_e^2\right)^{1/2}$$

Also error estimator should not overestimate the tolerance. This is the effectivity of a mesh element.

$$\eta_e \le \lambda_e \qquad (17)$$

It avoids unnecessary refinements. This property often can be ensured locally, but up to some constant and with respect to some domain of influence at the right hand size.

We define the ratio of estimated error and the tolerance as effectivity index. In particular, it is desired that the effectivity index approaches to one, as the exact error tends to zero.

5 Applications of anisotropic mesh

In the data fields of computational fluid dynamics, anisotropy arise from flow features such as rapid variations, or discontinuities, in pressure (shock waves), density (contact surfaces), velocity components (boundary layers, shear layers). As stated earlier, the information for controlling the improvements in the new mesh is summarized in a metric tensor, which is stored as a part of the old mesh, which we will refer to as the control mesh. Moreover, the tensor data is stored as the size of the scales for the triangles, the orientation of this length scale and a stretching ratio, giving the ratio of the longer to the shorter length scale[5].

Figure shows an example of an unstructured, anisotropic mesh generated to support the computation of the flow around an aerfoil.

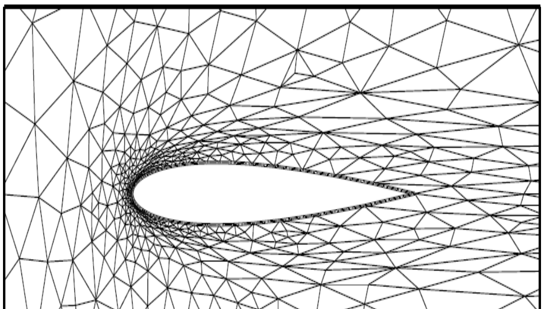

Fig2. Anisotropic mesh for flow past an aerofoil , Courtesy: M-G Vallet ,[7]

Fig 3. shows initial mesh obtained by Delaunay triangulation algorithm. The adapted mesh obtained by anisotropic mesh is shown in figure 4. for solving the simple elliptic PDE.

The poisson's problem is given by,

$$-\Delta u = 1 \text{ on } D, \qquad u = 0 \text{ on } \partial D,$$

Where Δ is the laplace's differential operator, D is the 'L' shaped domain. ∂D is the boundary of the domain.

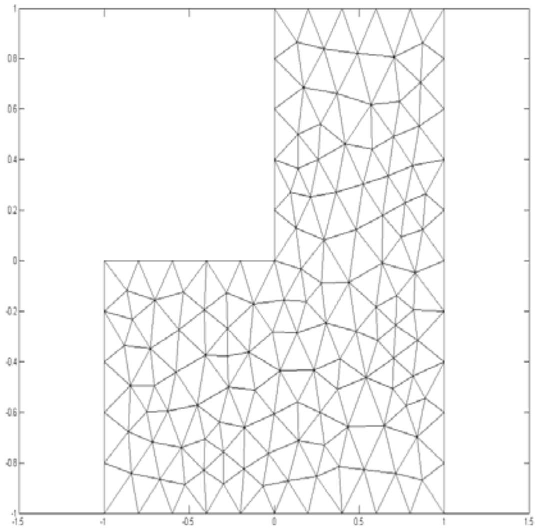

Fig 3. Initial mesh

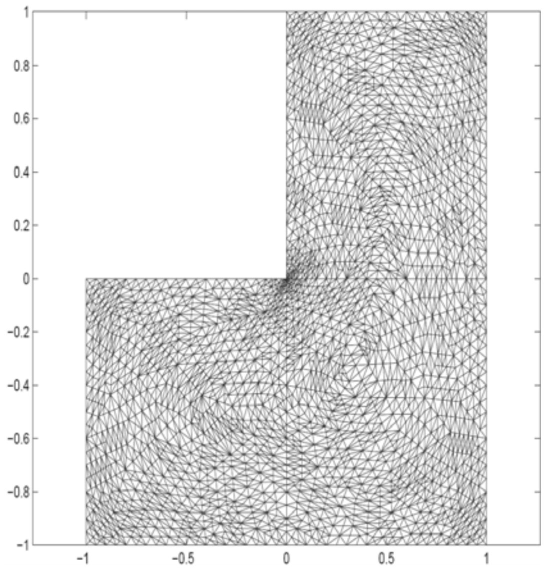

Fig 4. Anisotropic refinement of 'L' shape domain

6. Conclusion

In this paper, theoretical knowledge adapted for an anisotropic adaptation mesh scheme is presented. Construction of metric tensor and role of the same in generating anisotropic mesh is presented in detail. Mathematical expression for edge length of anisotropic triangle has been carried out. Mesh properties like reliability and effectivity are being derived mathematically. Finally the adaptation scheme is utilized in MATLAB to solve Poisson's problem.

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