Numerical Methods for Compressible Boundary Flow Stability

DONG Xue-zhi Tan Chun-qing
Institute of Engineering Thermophysics of Chinese Academy of Science
Beijing, 100080, China
E-mail: dongxuezhi@mail.etp.ac.cn

Keywords: Linear Stability, Eigenvalue, PSE method, Coutte Shear Flow

Abstract

Methods for the solution of linear stability for compressible boundary layers are developed. Both the global and local methods for stability analysis are used. Both methods are use in solution of Coutte shear flow and the results are analysis and compare. Some well-known conclusions of Coutte flow are proved by these methods again.

Introduction

The transition to turbulence of the boundary plays an important role in airfoil design. The linear stability analysis of a typical fluid flow is the general tools in flow stability analysis. Transition to turbulence in a boundary layer is a nonlinear process, but the way in which nonlinearity develops depends on the linear behavior of initially small disturbance. For the incompressible flow and used the parallel-flow approximation, the solution of boundary flow stability is leaded to the definition of Orr-Sommerfeld eigenvalue problem. In incompressible parallel flow, the Squire theory is used to translating three-dimension stability problem into two-dimension problem.

In compressible boundary flow, the O-S equation is inapplicable. The disturbance terms aren't only velocity and density, but also temperature, viscosity, thermal conductivity and specific heat. And the Squire theory doesn't adapt to compressible flow. The parabolized stability equations (PSE) approach is more accurate and consistent than the classical O-S theory. Using finite-difference discretization, the differential equations in PSE are reduced to linear algebraic equations. The global eigenvalues can be obtained by solving the characteristic determinant of a generalized eigenvalue problem. QR algorithm is used to compute all the eigenvalues. The number of solutions is relative to number of grid points, and many of "spurious" eigenvalues appear for grid points. A "spurious" eigenvalue is one which is not an eigenvalue of the differential operator. When filtering out these "spurious" eigenvalues, the global problem is solved. If a guess of the eigenvalues is available, then the eigenvalue may be purified by a local eigenvalue search procedure involving matrix inversion and Neton iteration. [1][2]

This paper develops a global method of solution to the compressible boundary flow stability. The method exploits the tridiagonal structure associated with a finite-difference representation of the operator involved, to automatically filter out spurious eigenvalues and calculate the non-spurious eigenvectors. [3]The unstable one of all true eigenvalues is chose, and is picked as the guess value of local method. The discretization which is used in local method is more accurate than that in global method. Matrix inversion and Neton iteration are used to solution local eigenvalue, and to compute the corresponding eigenvector. [4][5]

Equation

Compressible Linear Stability Equations

The Navier-Stokes equations governing the flow of a viscous compressible ideal gas are

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho v)=0 \tag{1}$$

$$\rho\left[\frac{\partial v}{\partial t}+(v\cdot\nabla)v\right]=-\nabla p+\frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)+\delta_{ij}\lambda\nabla\cdot v\right] \tag{2}$$

$$\rho c_p\left[\frac{\partial T}{\partial t}+(v\cdot\nabla)T\right]=\left[\frac{\partial p}{\partial t}+(v\cdot\nabla)p\right]+\nabla\cdot(k\nabla T)+\Phi \tag{3}$$

$$p=\rho RT \tag{4}$$

Where ρ is the density, p the pressure, v the velocity vector, T edge temperature, μ the first coefficient of viscosity, λ the second coefficient of viscosity, c_p the specific heat, k the thermal conductivity x_1 , x_2 x_3 is the three diction of Cattesian coordinates. The viscous dissipation Φ is given as

$$\Phi=\frac{\partial u_i}{\partial x_j}\left\{\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)+\delta_{ij}\lambda(\nabla\cdot v)\right\} \tag{5}$$

In this case we use dimensionless quantities,

$x_j^*=x_j/l$, $t^*=tu_e/l$, $v^*=v/u_e$, $T^*=T/T_e$, $\rho^*=\rho/\rho_e$, $p^*=p/(\rho_e u_e^2)$, $\mu^*=\mu/\mu_e$, $k^*=k/k_e$, $c_p^*=c_p/c_{pe}$, $\lambda^*=\lambda/\lambda_e$

where l is reference length, l/u_e time, u_e the mainstream edge velocity, ρ_e edge density, T_e edge temperature, μ_e edge viscosity, k_e edge thermal conductivity, c_{pe} edge specific heat, λ_e edge the second coefficient of viscosity. The variable with superscript "*" is the nondimensional.

We use Cattesion coordinates x^*, y^*, z^*, where x^* is the streamwise direction, y^* the spanwise direction and z^* is normal to the solid boundary. And the victor correspondence of three diction are u^*, v^*, w^*. The nondimensional Navier-Stokes equations is

$$\frac{\partial \rho^*}{\partial t^*}+\frac{\partial(\rho^* u^*)}{\partial x^*}+\frac{\partial(\rho^* v^*)}{\partial y^*}+\frac{\partial(\rho^* w^*)}{\partial z^*}=0 \tag{6}$$

$$\rho^*\frac{Du^*}{Dt^*}=-\frac{\partial p^*}{\partial x^*}+\frac{1}{\mathrm{Re}}\left\{\frac{\partial}{\partial x^*}\left(2\mu^*\frac{\partial u}{\partial x^*}+l_0\lambda^*\nabla\cdot V\right)+\frac{\partial}{\partial y^*}\left[\mu^*\left(\frac{\partial u^*}{\partial y^*}+\frac{\partial v^*}{\partial x^*}\right)\right]+\frac{\partial}{\partial z^*}\left[\mu^*\left(\frac{\partial w^*}{\partial x^*}+\frac{\partial u^*}{\partial z^*}\right)\right]\right\} \tag{7}$$

$$\rho^*\frac{Dv^*}{Dt^*}=-\frac{\partial p^*}{\partial y^*}+\frac{1}{\mathrm{Re}}\left(\frac{\partial}{\partial y^*}\left(2\mu^*\frac{\partial v}{\partial y^*}+l_0\lambda^*\nabla\cdot V\right)+\frac{\partial}{\partial x^*}\left[\mu^*\left(\frac{\partial v^*}{\partial x^*}+\frac{\partial u^*}{\partial y^*}\right)\right]+\frac{\partial}{\partial z^*}\left[\mu^*\left(\frac{\partial v^*}{\partial z^*}+\frac{\partial w^*}{\partial y^*}\right)\right]\right\} \tag{8}$$

$$\rho^*\frac{Dv^*}{Dt^*}=-\frac{\partial p^*}{\partial y^*}+\frac{1}{\mathrm{Re}}\left\{\frac{\partial}{\partial y^*}\left(2\mu^*\frac{\partial v}{\partial y^*}+l_0\lambda^*\nabla\cdot V\right)+\frac{\partial}{\partial x^*}\left[\mu^*\left(\frac{\partial v^*}{\partial x^*}+\frac{\partial u^*}{\partial y^*}\right)\right]+\frac{\partial}{\partial z^*}\left[\mu^*\left(\frac{\partial v^*}{\partial z^*}+\frac{\partial w^*}{\partial y^*}\right)\right]\right\} \tag{9}$$

$$\rho^* c_p^*\frac{DT^*}{Dt^*}=Ec\frac{Dp^*}{Dt^*}+\frac{1}{\mathrm{Re}\,\mathrm{Pr}}\nabla^*\cdot(k^*\nabla^* T^*)+\frac{Ec}{\mathrm{Re}}\Phi^* \tag{10}$$

$$\gamma ECp^*=(\gamma-1)\rho^* T^* \tag{11}$$

Where $l_0=\lambda_e/\mu_e$ is viscosity ratio, $Ec=u_e^2/(c_{pe}T_e)$ is Eckert number, $\mathrm{Pr}=\mu_e c_{pe}/k_e$ is Prandtl number, $\gamma=c_p/c_v$ is specific heat ratio. In ideal gas, $c_p^*=1$. Where

$$\Phi^*=\frac{\partial u_i^*}{\partial x_j^*}\left\{\mu^*\left(\frac{\partial u_i^*}{\partial x_j^*}+\frac{\partial u_j^*}{\partial x_i^*}\right)+\delta_{ij}l_0\lambda^*(\nabla^*\cdot v^*)\right\} \tag{12}$$

The instantaneous nondimensional values can be represented as the sum of a mean and a fluctuation quantity,

$u^*=\overline{U}+u'$, $v^*=\overline{V}+v'$, $w^*=\overline{W}+w'$, $p^*=\overline{P}+p'$, $T^*=\overline{T}+T'$, $\rho^*=\overline{\rho}+\rho'$, $\mu^*=\overline{\mu}+\mu'$, $\lambda^*=\overline{\lambda}+\lambda'$, $k^*=\overline{k}+k'$

Substituting these into the nondimensional equation and dropping the mean quantities, yields the linearized perturbation equation

$$\frac{\partial \rho'}{\partial t^*}+\frac{\partial(\overline{\rho}u'+\rho'\overline{u})}{\partial x^*}+\frac{\partial(\overline{\rho}v'+\rho'\overline{v})}{\partial y^*}+\frac{\partial(\overline{\rho}w'+\rho'\overline{w})}{\partial z^*}=0 \tag{13}$$

$$\overline{\rho}\left(\frac{\partial u'}{\partial t^*}+\overline{U}\frac{\partial u'}{\partial x^*}+u'\frac{\partial\overline{U}}{\partial x^*}+\overline{V}\frac{\partial u'}{\partial y^*}+v'\frac{\partial\overline{U}}{\partial y^*}+\overline{W}\frac{\partial u'}{\partial z^*}+w'\frac{\partial\overline{U}}{\partial z^*}\right)+\rho'\left(\overline{U}\frac{\partial\overline{U}}{\partial x^*}+\overline{V}\frac{\partial\overline{U}}{\partial y^*}+\overline{W}\frac{\partial\overline{U}}{\partial z^*}\right)=-\frac{\partial p'}{\partial x^*}+$$
$$\frac{1}{\mathrm{Re}}\left\{\frac{\partial}{\partial x^*}\left[\left(2\mu'+l_0\lambda'\right)\frac{\partial\overline{U}}{\partial x^*}+l_0\lambda'\frac{\partial\overline{V}}{\partial y^*}+l_0\lambda'\frac{\partial\overline{W}}{\partial z^*}+\left(2\overline{\mu}+l_0\overline{\lambda}\right)\frac{\partial u'}{\partial x^*}+l_0\overline{\lambda}\frac{\partial v'}{\partial y^*}+l_0\overline{\lambda}\frac{\partial w'}{\partial z^*}\right]+\frac{\partial}{\partial y^*}\left[\overline{\mu}\left(\frac{\partial u'}{\partial y^*}+\frac{\partial v'}{\partial x^*}\right)+\mu'\left(\frac{\partial\overline{U}}{\partial y^*}+\frac{\partial\overline{V}}{\partial x^*}\right)\right]+\frac{\partial}{\partial z^*}\left[\overline{\mu}\left(\frac{\partial w'}{\partial x^*}+\frac{\partial u'}{\partial z^*}\right)+\mu'\left(\frac{\partial\overline{W}}{\partial x^*}+\frac{\partial\overline{U}}{\partial z^*}\right)\right]\right\} \tag{14}$$

$$\overline{\rho}\left(\frac{\partial v'}{\partial t^*}+\overline{U}\frac{\partial v'}{\partial x^*}+u'\frac{\partial\overline{V}}{\partial x^*}+\overline{V}\frac{\partial v'}{\partial y^*}+v'\frac{\partial\overline{V}}{\partial y^*}+\overline{W}\frac{\partial v'}{\partial z^*}+w'\frac{\partial\overline{V}}{\partial z^*}\right)+\rho'\left(\overline{U}\frac{\partial\overline{V}}{\partial x^*}+\overline{V}\frac{\partial\overline{V}}{\partial y^*}+\overline{W}\frac{\partial\overline{V}}{\partial z^*}\right)=-\frac{\partial p'}{\partial y^*}+$$
$$\frac{1}{\mathrm{Re}}\left\{\frac{\partial}{\partial x^*}\left[\overline{\mu}\left(\frac{\partial u'}{\partial y^*}+\frac{\partial v'}{\partial x^*}\right)+\mu'\left(\frac{\partial\overline{U}}{\partial y^*}+\frac{\partial\overline{V}}{\partial x^*}\right)\right]+\frac{\partial}{\partial y^*}\left[l_0\lambda'\frac{\partial\overline{U}}{\partial x^*}+\left(2\mu'+l_0\lambda'\right)\frac{\partial\overline{V}}{\partial y^*}+l_0\lambda'\frac{\partial\overline{W}}{\partial z^*}+l_0\overline{\lambda}\frac{\partial u'}{\partial x^*}+\left(2\overline{\mu}+l_0\overline{\lambda}\right)\frac{\partial v'}{\partial y^*}+l_0\overline{\lambda}\frac{\partial w'}{\partial z^*}\right]+\frac{\partial}{\partial z^*}\left[\overline{\mu}\left(\frac{\partial v'}{\partial z^*}+\frac{\partial w'}{\partial y^*}\right)+\mu'\left(\frac{\partial\overline{V}}{\partial z^*}+\frac{\partial\overline{W}}{\partial y^*}\right)\right]\right\} \tag{15}$$

$$\overline{\rho}\left(\frac{\partial w'}{\partial t^*}+\overline{U}\frac{\partial w'}{\partial x^*}+u'\frac{\partial\overline{W}}{\partial x^*}+\overline{V}\frac{\partial w'}{\partial y^*}+v'\frac{\partial\overline{W}}{\partial y^*}+\overline{W}\frac{\partial w'}{\partial z^*}+w'\frac{\partial\overline{W}}{\partial z^*}\right)+\rho'\left(\overline{U}\frac{\partial\overline{W}}{\partial x^*}+\overline{V}\frac{\partial\overline{W}}{\partial y^*}+\overline{W}\frac{\partial\overline{W}}{\partial z^*}\right)=-\frac{\partial p'}{\partial z^*}+$$
$$\frac{1}{\mathrm{Re}}\left\{\frac{\partial}{\partial x^*}\left[\overline{\mu}\left(\frac{\partial w'}{\partial x^*}+\frac{\partial u'}{\partial z^*}\right)+\mu'\left(\frac{\partial\overline{W}}{\partial x^*}+\frac{\partial\overline{U}}{\partial z^*}\right)\right]+\frac{\partial}{\partial y^*}\left[\overline{\mu}\left(\frac{\partial w'}{\partial y^*}+\frac{\partial v'}{\partial z^*}\right)+\mu'\left(\frac{\partial\overline{W}}{\partial y^*}+\frac{\partial\overline{V}}{\partial z^*}\right)\right]+\frac{\partial}{\partial z^*}\left[l_0\lambda'\frac{\partial\overline{U}}{\partial x^*}+l_0\lambda'\frac{\partial\overline{V}}{\partial y^*}+\left(2\mu'+l_0\lambda'\right)\frac{\partial\overline{W}}{\partial z^*}+l_0\overline{\lambda}\frac{\partial u'}{\partial x^*}+l_0\overline{\lambda}\frac{\partial v'}{\partial y^*}+\left(2\overline{\mu}+l_0\overline{\lambda}\right)\frac{\partial w'}{\partial z^*}\right]\right\} \tag{16}$$

$$\overline{\rho}\left[\frac{\partial T'}{\partial t^*}+\overline{U}\frac{\partial T'}{\partial x^*}+u'\frac{\partial\overline{T}}{\partial x^*}+\overline{V}\frac{\partial T'}{\partial y^*}+v'\frac{\partial\overline{T}}{\partial y^*}+\overline{W}\frac{\partial T'}{\partial z^*}+w'\frac{\partial\overline{T}}{\partial z^*}\right]+\rho'\left(\overline{U}\frac{\partial\overline{T}}{\partial x^*}+\overline{V}\frac{\partial\overline{T}}{\partial y^*}+\overline{W}\frac{\partial\overline{T}}{\partial z^*}\right)=$$
$$Ec\overline{\rho}\left[\frac{\partial p'}{\partial t^*}+\overline{U}\frac{\partial p'}{\partial x^*}+u'\frac{\partial\overline{p}}{\partial x^*}+\overline{V}\frac{\partial p'}{\partial y^*}+v'\frac{\partial\overline{p}}{\partial y^*}+\overline{W}\frac{\partial p'}{\partial z^*}+w'\frac{\partial\overline{p}}{\partial z^*}\right]+Ec\rho'\left(\overline{U}\frac{\partial\overline{p}}{\partial x^*}+\overline{V}\frac{\partial\overline{p}}{\partial y^*}+\overline{W}\frac{\partial\overline{p}}{\partial z^*}\right)$$
$$+\frac{1}{\mathrm{Re}\,\mathrm{Pr}}\left[\frac{\partial}{\partial x^*}\left(\overline{k}\frac{\partial T'}{\partial x^*}+k'\frac{\partial\overline{T}}{\partial x^*}\right)+\frac{\partial}{\partial y^*}\left(\overline{k}\frac{\partial T'}{\partial y^*}+k'\frac{\partial\overline{T}}{\partial y^*}\right)+\frac{\partial}{\partial z^*}\left(\overline{k}\frac{\partial T'}{\partial z^*}+k'\frac{\partial\overline{T}}{\partial z^*}\right)\right]$$
$$+\frac{Ec}{\mathrm{Re}}\left\{\overline{\mu}\left[4\left(\frac{\partial\overline{U}}{\partial x^*}\frac{\partial u'}{\partial x^*}+\frac{\partial\overline{V}}{\partial y^*}\frac{\partial v'}{\partial y^*}+\frac{\partial\overline{W}}{\partial z^*}\frac{\partial w'}{\partial z^*}\right)+2\left(\frac{\partial\overline{V}}{\partial x^*}+\frac{\partial\overline{U}}{\partial y^*}\right)\left(\frac{\partial v'}{\partial x^*}+\frac{\partial u'}{\partial y^*}\right)+2\left(\frac{\partial\overline{W}}{\partial y^*}+\frac{\partial\overline{V}}{\partial z^*}\right)\left(\frac{\partial w'}{\partial y^*}+\frac{\partial v'}{\partial z^*}\right)+2\left(\frac{\partial\overline{U}}{\partial z^*}+\frac{\partial\overline{W}}{\partial x^*}\right)\left(\frac{\partial u'}{\partial z^*}+\frac{\partial w'}{\partial x^*}\right)\right]\right.$$
$$+2\overline{\lambda}l_0\left(\frac{\partial\overline{U}}{\partial x^*}+\frac{\partial\overline{V}}{\partial y^*}+\frac{\partial\overline{W}}{\partial z^*}\right)\left(\frac{\partial u'}{\partial x^*}+\frac{\partial v'}{\partial y^*}+\frac{\partial w'}{\partial z^*}\right)+\mu'\left[2\left(\frac{\partial\overline{U}}{\partial x^*}\right)^2+2\left(\frac{\partial\overline{V}}{\partial y^*}\right)^2+2\left(\frac{\partial\overline{W}}{\partial z^*}\right)^2\right.$$
$$\left.\left.+\left(\frac{\partial\overline{V}}{\partial x^*}+\frac{\partial\overline{U}}{\partial y^*}\right)^2+\left(\frac{\partial\overline{W}}{\partial y^*}+\frac{\partial\overline{V}}{\partial z^*}\right)^2+\left(\frac{\partial\overline{U}}{\partial z^*}+\frac{\partial\overline{W}}{\partial x^*}\right)^2\right]+\lambda' l_0\left(\frac{\partial\overline{U}}{\partial x^*}+\frac{\partial\overline{V}}{\partial y^*}+\frac{\partial\overline{W}}{\partial z^*}\right)^2\right\} \tag{17}$$

$$\gamma ECp'=(\gamma-1)\left(\overline{\rho}T'+\rho'\overline{T}\right) \tag{18}$$

Using Stokes assumption

$$\lambda/\mu=const=-2/3 \tag{19}$$

So

$$\lambda^*=\mu^*,\ l_0=-2/3 \tag{20}$$

Assumed $\mu=\mu(T)$, $\lambda=\lambda(T)$, $k=k(T)$, they are all function of temperature. So we can write

$$\mu'=\frac{d\overline{\mu}}{d\overline{T}}T',\ \lambda'=\frac{d\overline{\lambda}}{d\overline{T}}T',\ k'=\frac{d\overline{k}}{d\overline{T}}T' \tag{21}$$

n boundary layer assumption

$$\frac{\partial\overline{p}}{\partial y^*}=0,\ \overline{U}=\overline{U}(y^*),\ \overline{W}=\overline{W}(y^*),\ \overline{T}=\overline{T}(y^*),$$
$$\overline{\rho}=\overline{\rho}(y^*),\ \overline{V}=0 \tag{22}$$

So the mean equation of state is

$$1=\overline{\rho}\overline{T} \tag{23}$$

So

$$\rho' = \frac{\gamma}{\gamma-1} Ec \frac{p'}{\overline{T}} - \frac{T'}{\overline{T}^2} = f_1(\overline{T})p' + f_2(\overline{T})T' \qquad (24)$$

We may now assume that the velocity, pressure, and temperature fluctuations may be represented by a harmonic wave of form

$$[u',v',w',p',T'] = [\tilde{u}(y^*),\tilde{v}(y^*),\tilde{w}(y^*),\tilde{p}(y^*),\tilde{T}(y^*)]\exp[i(\alpha x^* + \beta y^* + \omega t^*)] \qquad (25)$$

Where α , β are the wavemumbers and ω is the frequency which, in general, are all complex.

Substitution Eq (25) into Eq (13)-(17), it can shown that the linear disturbance satisfy the following system of ordinary differential equations

$$\alpha\overline{\rho}\tilde{u} + \left(\frac{d\overline{\rho}}{dy^*} + \overline{\rho}\frac{d}{dy^*}\right)\tilde{v} + \beta\overline{\rho}\tilde{w} + (\alpha\overline{U} + \beta\overline{W} - \omega) f_1\tilde{p} + (\alpha\overline{U} + \beta\overline{W} - \omega) f_2\tilde{T} = 0 \qquad (26)$$

$$\left[\frac{\overline{\mu}}{\mathrm{Re}}\frac{d^2}{dy^{*2}} + \frac{1}{\mathrm{Re}}\frac{d\overline{\mu}}{dy^*}\frac{d}{dy^*} + \frac{\overline{\mu}}{\mathrm{Re}}\left(\frac{4}{3}\alpha^2 + \beta^2\right) - \overline{\rho}\left(\alpha\overline{U} + \beta\overline{W} - \omega\right)\right]\tilde{u} + \left(\frac{1}{3}\frac{\alpha\overline{\mu}}{\mathrm{Re}}\frac{d}{dy^*} - \frac{\alpha}{\mathrm{Re}}\frac{d\overline{\mu}}{dy^*} - \overline{\rho}\frac{d\overline{U}}{dy^*}\right)\tilde{v} + \frac{\alpha\beta\overline{\mu}}{3\mathrm{Re}}\tilde{w} - \alpha\tilde{p} + \left(\frac{1}{\mathrm{Re}}\frac{d\overline{U}}{dy^*}\frac{d^2\overline{\mu}}{d\overline{T}^2}\frac{d}{dy^*} + \frac{1}{\mathrm{Re}}\frac{d^2\overline{U}}{dy^{*2}}\frac{d\overline{\mu}}{d\overline{T}}\right)\tilde{T} = 0 \qquad (27)$$

$$\left(\frac{1}{3}\frac{\alpha\overline{\mu}}{\mathrm{Re}}\frac{d}{dy^*} - \frac{2}{3}\frac{1}{\mathrm{Re}}\frac{d\overline{\mu}}{dy^*}\right)\tilde{u} + \left[\frac{4\overline{\mu}}{3\mathrm{Re}}\frac{d^2}{dy^{*2}} + \frac{4\overline{\mu}}{3\mathrm{Re}}\frac{d\overline{\mu}}{dy^*}\frac{d}{dy^*} + \frac{\overline{\mu}\left(\alpha^2+\beta^2\right)}{\mathrm{Re}} - \overline{\rho}\left(\alpha\overline{U} + \beta\overline{W} - \omega\right)\right]\tilde{v} + \left(\frac{1\beta\overline{\mu}}{3\mathrm{Re}}\frac{d}{dy^*} - \frac{2\beta}{3\mathrm{Re}}\frac{d\overline{\mu}}{dy^*}\right)\tilde{w} + \left(-\frac{d}{dy^*}\right)\tilde{p} + \left(\frac{\alpha}{\mathrm{Re}}\frac{d\overline{U}}{dy^*} + \frac{\beta}{\mathrm{Re}}\frac{d\overline{W}}{dy^*}\right)\frac{d\overline{\mu}}{d\overline{T}}\tilde{T} = 0 \qquad (28)$$

$$\frac{\alpha\beta\overline{\mu}}{3\mathrm{Re}}\tilde{u} + \left(\frac{1}{3}\frac{\beta\overline{\mu}}{\mathrm{Re}}\frac{d}{dy^*} + \frac{\beta}{\mathrm{Re}}\frac{d\overline{\mu}}{dy^*} - \overline{\rho}\frac{d\overline{W}}{dy^*}\right)\tilde{v} + \left[\frac{\overline{\mu}}{\mathrm{Re}}\frac{d^2}{dy^{*2}} + \frac{1}{\mathrm{Re}}\frac{d\overline{\mu}}{dy^*}\frac{d}{dy^*} + \frac{\overline{\mu}}{\mathrm{Re}}\left(\alpha^2 + \frac{4}{3}\beta^2\right) - \overline{\rho}\left(\alpha\overline{U} + \beta\overline{W} - \omega\right)\right]\tilde{w} - \beta\tilde{p} + \left(\frac{1}{\mathrm{Re}}\frac{d\overline{W}}{dy^*}\frac{d^2\overline{\mu}}{d\overline{T}^2}\frac{d}{dy^*} + \frac{1}{\mathrm{Re}}\frac{d^2\overline{W}}{dy^{*2}}\frac{d\overline{\mu}}{d\overline{T}}\right)\tilde{T} = 0 \qquad (29)$$

$$\frac{2\alpha\overline{\mu}Ec}{\mathrm{Re}}\frac{d\overline{U}}{dy^*}\tilde{u} + \left[\frac{2\overline{\mu}Ec}{\mathrm{Re}}\left(\alpha\frac{d\overline{U}}{dy^*} + \beta\frac{d\overline{W}}{dy^*}\right) + Ec\overline{\rho}\frac{d\overline{p}}{dy^*} - \frac{d\overline{\rho}}{dy^*}\right]\tilde{v} + \frac{2\beta\overline{\mu}Ec}{\mathrm{Re}}\frac{d\overline{W}}{dy^*}\tilde{w} + Ec\overline{\rho}\left(\alpha\overline{U} + \beta\overline{W} - \omega\right)\tilde{p} + \left\{\frac{\overline{k}}{\mathrm{Re}\,\mathrm{Pr}}\left(\alpha^2 + \beta^2\right) + \frac{1}{\mathrm{Re}\,\mathrm{Pr}}\left[\overline{k}\frac{d^2}{dy^{*2}} + \left(\frac{d\overline{T}}{dy^*}\frac{d^2\overline{k}}{d\overline{T}^2} + \frac{d\overline{k}}{d\overline{T}}\right)\frac{d}{dy^*} + \frac{d^2\overline{T}}{dy^{*2}}\right] + \frac{Ec}{\mathrm{Re}}\frac{d\overline{\mu}}{d\overline{T}}\left[\left(\frac{d\overline{U}}{dy^*}\right)^2 + \left(\frac{d\overline{W}}{dy^*}\right)^2\right] - \overline{\rho}\left(\alpha\overline{U} + \beta\overline{W} - \omega\right)\right\}\tilde{T} = 0 \qquad (30)$$

Assumed $\mu = \mu(T)$ only relative with temperature. In Sutherland formula

$$\mu^* = T^{*3/2}\,\frac{1 + S/T_e}{T^* + S/T_e} \qquad (31)$$

Where coefficient S=110.4 in air.

For air, if the temperature changes form 0 degrees centigrade to 1000 degrees centigrade, the viscosity μ increases 140 percent, and the thermal conductivity k increases 190 percent, the ratio k/μ only increases 20 percent. So assumed $k/\mu = const$. This assumption is appropriate and general in boundary theory. Thus $k^* = \mu^*$.

In Eq. (26)- Eq. (30), the operator d/dy^* and d^2/dy^{*2} is linear differential operator. If ω , β are assumed to be real and presetting value. α is complain. So if the imaginary part of α is positive, the disturbance decays exponentially in x direction, so the flow is stability; if the imaginary part of α is negative, the disturbance grows exponentially in x direction, so the flow isn't stablilty; if he imaginary part of α is zero, the disturbance neither grows nor decays.

Eq. (24)- Eq. (28) can be shown as linear disturbance equation

$$\left(A\alpha^2 + B\alpha + C\right)\Phi = 0 \qquad (32)$$

Where $\Phi = \left(\tilde{u},\tilde{v},\tilde{w},\tilde{p},\tilde{t}\right)'$

That is an eigenvalue problem, which can be solute by discretization.

<u>Discretization of the Eigenvalue Problem</u>

Single domain spectral (SDSP) collocation method and directly numerical difference are used in differential operator's discretization.

In spectral method, we use Nth order Chebyshev polynomials defined on the interval [-1, 1], where the collocation points

$\xi_j = \cos(\pi j/N); \quad j = 1,1,\ldots,N$.

The first derivative of $\phi(\xi_j)$ may be written as

$$d\phi/d\xi = E\phi \qquad (33)$$

Where N+1 order square matrix E are the discretization operator given as

$$E_{jk} = \frac{c_j}{c_k}\frac{(-1)^{k+j}}{x_j - x_k}; \; j \neq k \qquad (34)$$

$$E_{jj} = -\frac{x_j}{2\left(1 - x_j^2\right)} \qquad (35)$$

$$E_{00} = \frac{2N^2 + 1}{6} = -E_{NN} \qquad (36)$$

Where $c_0 = c_N = 2; c_k = 1$.

So $d^2\phi/d\xi^2 = EE\phi = (E2)\phi$, where matrix $E2$ is the second order differential operator's discretization matrix.

In directly numerical difference,

$$d\phi/dy\big|_j = \left(-3\phi_{j-3} + 27\phi_{j-2} - 135\phi_{j-1} + 135\phi_{j+1} - 27\phi_{j+2} + 3\phi_{j+3}\right)/180 \qquad (37)$$

$$d^2\phi/dy^2\big|_j = \left(2\phi_{j-3} - 27\phi_{j-2} + 270\phi_{j-1} - 490\phi_j + 270\phi_{j+1} - 27\phi_{j+2} + 2\phi_{j+3}\right)/180 \qquad (38)$$

This difference is sixth order accuracy. We change the difference scheme in boundary for forward or backward scheme, which ensure the sixth order accuracy in boundary position. Then the first and second order differential operator can be discretized as two N+1 order square matrix, which is band matrix.

If the scaling factor for the transformation between physical and computational domains is given as

$$S_j = \partial\xi/\partial y\big|_j \,; \; j = 0,1,\ldots,N \qquad (39)$$

$$(S2)_j = \partial^2\xi/\partial y^2\big|_j \,; \; j = 0,1,\ldots,N \qquad (40)$$

Then the first derivative matrix F in the physical domain may be written as

$$F_{jk} = S_j E_{jk} \qquad (41)$$

And the second derivative matrix G is simply

$$G_{jk} = S_j^2 (E2)_{jk} + S_j (S2)_j E_{jk} \tag{42}$$

Net Point Division

The Gauss node point is used in net point division. If the computational domains define on the interval [-1, 1] and the physical domains define on the $\left[-y_{\max}, y_{\max}\right]$, the function between computational domains and physical domains define as

$$y = \tanh(a \cdot x)/b \tag{43}$$

$$a = 0.5\left[\ln(1+b) - \ln(1-b)\right] \tag{44}$$

Where b is the net factor, which belongs to the interval (0, 1). In low velocity flow, the critical layer is nearly the wall. As the Mach number increase, the critical layer moves away from the wall toward the edge of the boundary layer. So choosing suitable net factor is important for solution precision. As the Mach number increase, the fact may choose little.

Mean flow

The Couette shear flow is used in solution. The Couette shear flow is shown in Fig. 1. The distance of two plates is $2Y_{\max}$. The upper plate is uniform motion with velocity U_e, and the low plate is stationary. The temperature of upper plate is T_e, and that of lower plate is T_0. The pressure changes along X director only, and the pressure gradient is const. By the no-slip boundary conditions

$$u = U_e, T = T_e, (y = Y_{\max}) \tag{45}$$

$$u = 0, T = T_0, (y = Y_{\max}) \tag{46}$$

Because the boundary conditions are independent of x and z, $T = T(y)$ is the only function of y, and unique velocity component $u = u(y)$ is the only function of y.

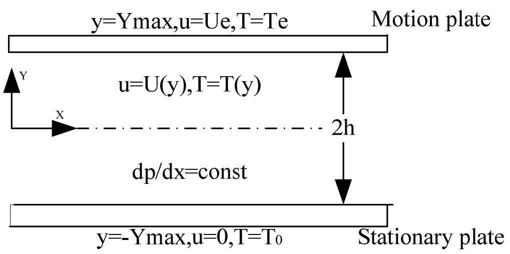

Fig.1 Coutter Shear Flow

For incompressible flow, the mean flow can be easily solve.

$$u = 0.5\left(y/y_{\max} + 1\right)U_e \tag{47}$$

$$T = \frac{1}{2}\left(T_e + T_0\right) + \frac{1}{2}\left(T_e - T_0\right)\left(y/y_{\max}\right) + T_e \frac{\Pr Ec}{8}\left[1-\left(y/y_{\max}\right)^2\right]$$
$$-T_e \frac{\Pr EcB}{6}\left[\left(y/y_{\max}\right) - \left(y/y_{\max}\right)^3\right] + T_e \frac{\Pr EcB^2}{12}\left[1-\left(y/y_{\max}\right)^4\right] \tag{48}$$

Where

$$B = -\frac{y_{\max}^2}{\mu U_e}\frac{dp}{dx} \tag{49}$$

B is nondimensional and rote of pressure gradient. If the pressure gradient is zero, B=0.

For incompressible flow, the density, the coefficient of viscosity, the specific heat and the thermal conductivity change with the pressure and temperature. If the pressure gradient is zero, the mean result can be written as an implicit function

$$y = C_1 \int_0^u \mu du - y_{\max} \tag{50}$$

$$\frac{1}{2}u^2 + \int_{T_0}^{T} k/\mu dT = C_2 u \tag{51}$$

C_1 and C_2 can be solved by boundary condition

$$u(y_{\max}) = U_e,\ T(y_{\max}) = T_e \tag{52}$$

Solution Equation

Substituting differential operator's discretization matrix and the mean result into linear disturbance equation, and considering the boundary condition:

$$\tilde{u} = \tilde{v} = \tilde{w} = 0,\ y = -1 \tag{53}$$

$$\tilde{u} = \tilde{v} = \tilde{w} = 0,\ y = 1 \tag{54}$$

A matrix equation is obtained.

$$D(\alpha) = C_0\alpha^2 + C_1\alpha + C_2 \tag{55}$$

$delt\left|D(\alpha)\right| = 0$ is the necessary and sufficient condition of equation having nontrivial solution. For solute this eigenvalue problem, both global and local eigenvalue methods are discussed.

global methods

The eigenvalue equation

$\det\left|C_0\alpha^2 + C_1\alpha + C_2\right| = 0$ can be changed as

$$\left\{\begin{bmatrix} -C_1 & -C_2 \\ I & 0 \end{bmatrix} - \alpha\begin{bmatrix} C_0 & 0 \\ 0 & I \end{bmatrix}\right\}\begin{Bmatrix} \alpha\Phi \\ \Phi \end{Bmatrix} = 0 \tag{56}$$

Where I is N+1 order identity matrix. If C_0 is idnvertible matrix, we can define a matrix

$$A = \begin{bmatrix} -C_0^{-1}C_1 & -C_0^{-1}C_2 \\ I & 0 \end{bmatrix} \tag{57}$$

So the eigenvalue of matrix A is the same as the solution of eigevalue equation. The eigenvalue of matrix A can be solved by matrix inversion technique. In this solution, transfer is used

$$\alpha = -\left(\lambda - 1\right)/\left(\lambda + 1\right) \tag{58}$$

So the eigenvale equation change as

$$\det\left|(C_0 - C_1 + C_2)\lambda^2 + (-C_0 + C_2)\lambda + (C_0 + C_1 + C_2)\right| = 0 \tag{59}$$

The use of this method leads to the appearance of "spurious" eigenvalues for discretization. A "spurious" eigenvalue is one which is not an eigenvalue of the eigenvalue equation. An infinite-dimensional operator is characterized by a spectrum comprising a countably infinite number of discrete eigenvalues and/or a continuous spectrum of eigenvalues. The discrete eigenvalues is expectation. As the number of nodes is increased, one would expect its spectrum to tend more and more towards some portion of the discrete part of the infinite-dimensional spectrum. However, the continuous part of the spectrum and eigenvalues are admixture. Sometime both the stable and unstable spurious eigenvalues have such large magnitudes that they are easily distinguished from the true eigenvalues. However, this may not be the case in all problems. The simple filtering "spurious" eigenvalues methods are solution under different nod number and choose the near result.

local methods

The global method can be used to solve all of the eigenvales, but it spends much solution time. And global method can't solve eigenfunctions. When guess for the eigenvalue is available, local methods may be used both to the eigenvalue and compute the associated eigenfunctions. But a good first guess is required to assure convergence and the determination of eigensolutions. After low order global solution, we choose the smallest imaginary part of the result as the first guess. Then the high order local methods will be used.

Newton's method is applied to find the eigenvalue. The iterative formula may be written

$$\alpha_{k+1} = \alpha_k - 2f(\alpha_k) / \left\{ \left[f(\alpha_k) \right]^2 - f^{(1)}(\alpha_k) \right\} \quad (60)$$

Where

$$f(\alpha_k) = Trace\left[D^{-1}(\alpha_k) D^{(1)}(\alpha_k) \right] \quad (61)$$

$$f^{(1)}(\alpha_k) = Trace\left[D^{-1}(\alpha_k) D^{(2)}(\alpha_k) - \left(D^{-1}(\alpha_k) D^{(1)}(\alpha_k) \right)^2 \right] \quad (62)$$

$Trace[A]$ denotes the trace of matrix D. D^{-1} is the inverse of matrix D, and $D^{(1)}$、$D^{(2)}$ denote the first and second order derivative of D with respect to α.

To compute single eigenfunction, the power iterative method is used. The iterative formula may be written

$$Z^{(k)} = A_4 V^{(k-1)} \quad (63)$$

$$\lambda^{(k)} = \left\| Z^{(k)} \right\|_\infty = \underset{i=1}{\overset{N}{Max}} \left[\left| Z_i^{(k)} \right| \right] \quad (64)$$

$$V^{(k)} = Z^{(k)} / \lambda^{(k)} \quad (65)$$

The vector $V^{(0)}$ is random. When $\left|\lambda^{(k)}\right|$ small enough, the iterative is will be stop.

An Example Computation

Fig. 2 and Fig. 3 are shown the different of incompressible and compressible mean flow. Fig.2 is the velocity distributing, and Fig. 3 is the temperature distributing. We can find there are small different in mean velocity, but big different in mean temperature.

On the air, when the velocity of upper plate is 2.0 Mach, and the temperature of upper and lower plate are all 300K , , the eigenvalue is solved. To filter out "spurious" eigenvalue, N=50, N=60 and N=70 are all computed. Fig.4 is the eigenvalue spectrum, and the arrow is point the smallest image part of eigenvalues. Table 3.1 is shown the eigenvalue in different node.

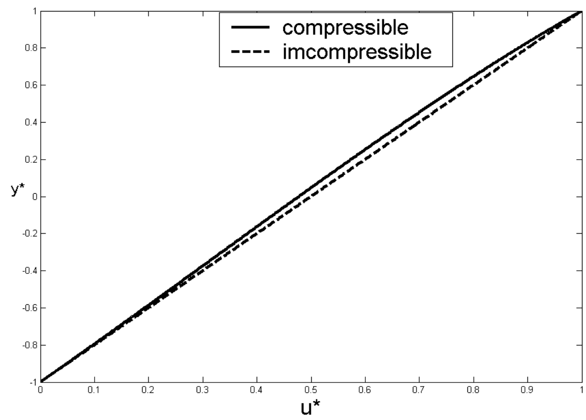

Fig.2 Velocity Distributing in Coutte Flow

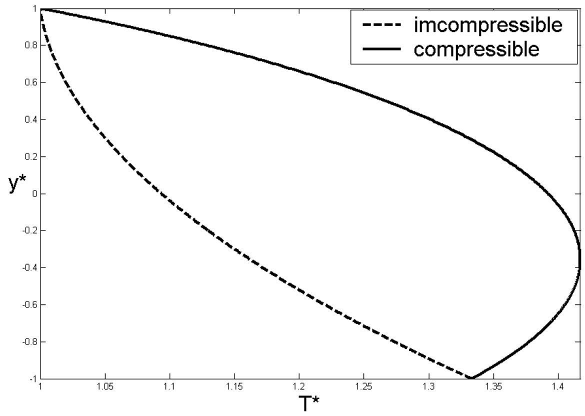

Fig.3 Temperature Distributing in Cutte Flow

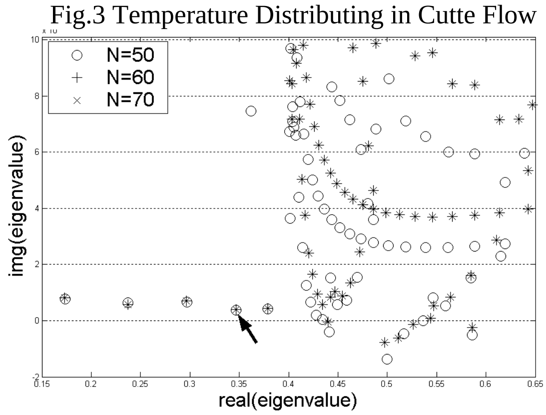

Fig. 4 Eigenvalue spectrum in complex space

Table 1 Eigenvalue with different node number

N	Eigenvalue
50	3.468e-001 +3.819e-005i
60	3.467e-001 +4.0257e-005i
70	3.467e-001 +4.250i
200	3.467e-001 +4.293i

N=200 is the result of local method. Fig.5 is shown the eigenfunction at N=200. Fig.6 is the eigenvector in Fourier space. We can find that the disturbance of and are big in stationary wall, but that of and are small in stationary wall. In the Fourie space, the Fourice are symmetry in two wall, and the coefficient is very big in tow wall. So the critical lay of Coutte happen in near of wall.

Fig.7 is shown the eigenfunction which the temperature of upper and lower plate are 300K and 400K. We can find the disturbance in hot wall is big than that in cool wall. The cool in wall make the flow stability.

Fig.8 is shown the change of Fouier coefficient with the Mach number. y-axis is the common logarithm of Fourier coefficient. In low velocity flow, the critical layer is nearly the wall. This is the same as the analysis in 1.3.

Conclusions

(1)We developed the eigenvalue solution methods for the compressible boundary layer stability equations. Both the global and local methods are used in solution. These methods are valuable methods in boundary stability problem.

(2)The critical lay of Coutte happen in near of wall, and move to the mid of flow with the increase of Mach number. The cool in wall make the flow stability.

References:

1) Herbert T., Bertolotti F.P., Analysis of nonparallel boundary layers. American Phys. Soc. 32, 1987.

2)Rama Govindarajan and R. Narasimha, Estimating amplitude ratios in boundary layer stability theory: a comparison between two approaches. J. Fluid Mech. (2001), vol. 439, pp. 403-412.

3)Sushil Shetty, An Efficient Numerical Method for Solving the Eigenvalue Problem Associated with a Quasi-Geostrophic Fluid Flow. MATH 221 Class Project, Spring 2003, University of California, Berkeley.

4)M. R. Malik, Numerical methods for hypersonic boundary layer stability. Journal of computational physics 86.376-413 (1990).

5)T. J. Bridges and P. J. Morris, Differential eigenvalue problems in which Parameter appears Nonlinearly. Journal of computational physics 55.437-460 (1984).