A Study on Two-Dimensional Positioning Algorithms Based on GPS Pseudorange Technique

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Abstract

In the paper, we have studied on algorithms for two-dimensional positioning based on GPS pseudorange Technique. First, the linearized state equation was mathematically derived based on GPS pseudorange technique. Second, the geometry model with respect to triangles formed using unit-vectors were proposed for investigation of land-based radio positioning. Finally, the corresponding mathematical formulations for DOP values and covariance matrix were designed for two-dimensional positioning.

Keyword

GPS, Pseudorange, DOP, Covariance matrix

Ⅰ. INTORDUCTION

The pseudorange positioning method, which has been initially applied to the GPS satellite navigation, is powerful to deal with the common bias of clock error between the transmitters and the user's receiver. It has the significant advantage to handle the navigation equation under the circumstance incoming signal from all transmitters in view.

In addition, it is well known that the positioning accuracy is deeply depended on the geometry influence. The geometry factor, GDOP(Geometry Dilution of Precision), which affects to the positioning accuracy, has mathematically defined and used for the three-dimensional satellite navigation[1].

Unfortunately, in case of two-dimensional positioning, there are no neat geometrical construction and the useful mathematical formulation from the logical concepts. Most of the previous materials involving research papers might have shown to be the lack of the systematic and integrated formulations for dealing with two-dimensional positioning with regard to the pseudorange technique[2]-[4].

In this paper, analytical algorithms based on mathematical steps are derived for two-dimensional positioning. Mainly, the linearized pseudorange equation and DOP values are investigated and discussed.

Ⅱ. GEOMETRIC CONSIDERATIONS FOR POSITIONING

A. Pseudorange Multiple Ranging Model

The unknown position P_n can be determined by measuring multiple ranges between P_n and the reference positions P_i.

In horizontal navigation, if two measurements are used to determine the fixing, then it is called the rho-rho method and rho-rho-rho in case of three measurements[5]. The range between the positions can be mathematically denoted by the length of the vector.

$$\rho_i = |X_i| \tag{1}$$

However, these positioning technique may not be clear for dealing with measurements having range error due to the erroneous synchronization of the receiver clock and the transmitter clocks.

The pseudorange multiple ranging is more flexible to apply to the erroneous positioning system because it can give the more precise solution in positioning and navigation as well.

That is, the following mathematical model, pseudorange, PR_i, can be used to solve the navigation systems which the measured ranges contain the range error $\delta\rho$ due to the common asynchronuous receiver clock.

$$PR_i = \rho_i + \delta\rho \tag{2}$$

Where, ρ_i is the true range between a receiver and *i-th* transmitter. Fig. 1 shows the concept of the pseudorange positioning fixing.

The solving three-dimensional coordinates of the unknown position, four pseudoranges are required. The three range measurements are needed in case of determining the coordinates of the horizontal position.

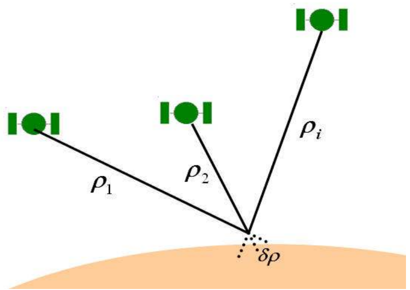

Fig. 1. The Concept of Pseudorange Positioning

B. Geometric Considerations on Positioning Accuracy

In case of three-dimensional satellite positioning, the GDOP value is deeply related to the volume of a tetrahedron formed using four unit-vectors point toward the GPS satellites. As it is the r.s.s(square root of sum of the squares) of the area of the 4 faces of the tetrahedron, divided by its volume[6].

In our investigation, it has been recognized that the tetrahedron represented by various materials seems to be unclear for ease understanding in proper coordinate. Therefore, we propose the meaningful tetrahedron shown in Fig. 2 with unit vector points toward the assumed broadcasting transmitters in east-north-up local coordinate. The larger the volume, the smaller the GDOP values. And then the low DOP value gives a better positioning precision because of the wider angular separation between the satellites used to calculate the receiver's position.

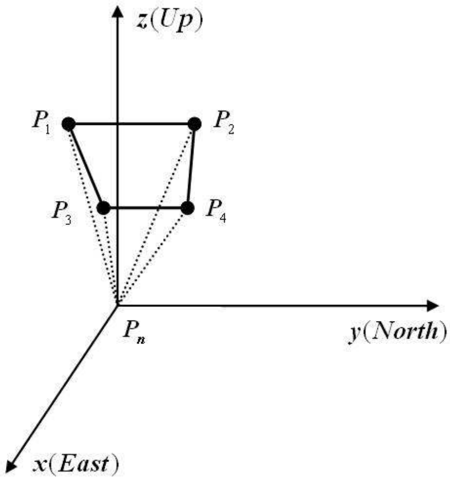

Fig. 2. Tetrahedron with Unit Vectors Points toward the GPS Satellites.

For two-dimensional GDOP, the area of triangles formed using three unit-vectors point toward the transmitters of the broadcasting stations may be considered unlike the tetrahedron. In this case, the receiver lies at the intersection of the circular lines of position that are centered on the transmitters.

There is some uncertainty, however, in the receiver's measurements, and the location of the range circles will be inexact and result in an error in computed position. These error depends on the geometry relating the receiver and transmitters.

Ⅲ. MATHEMATICAL FOR TWO-DIMEN-SIONAL POSITIONING BASED ON PSEUDORANGE MEASUREMENT

We develop the formulations with regard to two-dimensional positioning. As we mentioned previously, the PR method for determining three-dimensional position is very useful to compute the clock bias which is a term of unknown parameter in the linearized equation.

It means that an additional pseudorange measurement compared to the numbers of unknown of position coordinates is needed to construct the simultaneous equation. Therefor, three pseudorange measurements should be existed in order to solve the coordinates of two-dimensional position.

For simple derivation of the state equation, we consider N-transmitting stations on surface of earth or N-available navigation satellites in space.

And then the corresponding pseudoranges are available to formulate the equation.

With referring Fig. 1, let $P_n(x_u, y_u)$ and $P_i(x_i, y_i)$ are the receiver's position and the transmitting station's position, respectively. Then the pseudorange involving time offset, ct_u, can be expressed as the following set of equations.

$$
\begin{aligned}
\rho_1 &= \sqrt{(x_1 - x_u)^2 + (y_1 - y_u)^2} + ct_u \\
\rho_2 &= \sqrt{(x_2 - x_u)^2 + (y_2 - y_u)^2} + ct_u \\
\rho_3 &= \sqrt{(x_3 - x_u)^2 + (y_3 - y_u)^2} + ct_u \\
&\vdots \\
\rho_i &= \sqrt{(x_i - x_u)^2 + (y_i - y_u)^2} + ct_u \\
&= f(x_u, y_u, t_u)
\end{aligned} \tag{3}
$$

The above set of the nonlinear pseudorange equations should be linearized for determining position. According to the shown procedures, the linearization of the above nonlinear equation is taken.

In order to deal with Taylor series about the known approximate position $(x_y^{'}, y_u^{'})$, the corresponding approximate pseudorange might be represented by

$$
\begin{aligned}
\rho_i^{'} &= \sqrt{(x_i - x_u^{'})^2 + (y_i - y_u^{'})^2} + ct_u^{'} \\
&= f(x_u^{'}, y_u^{'}, t_u^{'})
\end{aligned} \tag{4}
$$

And then the function of the true receiver's position can be given with not only the offset but also time offset, δt_u.

$$f(x_u, y_u, t_u) = f(x_u^{'} + \delta x_u, y_u^{'} + \delta y_u, t_u^{'} + \delta t_u) \tag{5}$$

Therefore, the linearization works using Taylor series-expansion simply can be done.

G matrix is the components fo unit vector from the receiver to the indicated transmitter. Practically, *G* matrix can be obtained using the azimuth θ and elevation ϕ of satellites in case of three-dimensional positioning shown as follw[1]:

$$
G = \begin{bmatrix}
\cos\phi_1 \sin\theta_1 & \cos\phi_1 \cos\theta_1 & \sin\phi_1 & 1 \\
\cos\phi_2 \sin\theta_2 & \cos\phi_2 \cos\theta_2 & \sin\phi_2 & 1 \\
\cos\phi_3 \sin\theta_3 & \cos\phi_3 \cos\theta_3 & \sin\phi_3 & 1 \\
\vdots & \vdots & \vdots & \vdots \\
\cos\phi_n \sin\theta_n & \cos\phi_n \cos\theta_n & \sin\phi_n & 1
\end{bmatrix} \tag{6}
$$

However, the only azimuth from the receiver position to transmitters is used to formulate directional derivatives of the pseudorange range error. And then the azimuth could be translated into the horizontal plane coordinate frame which is with respect to the east and the north without the up. Therefore, if the third column of elevation term, ϕ, is deleted and the row deleted corresponding to the unused transmitter of the station in above G matrix. With zero degree-elevation, the resulting matrix for N-transmitters can be formulated as follow:

$$
G = \begin{bmatrix}
\sin\theta_1 & \cos\theta_1 & 1 \\
\sin\theta_2 & \cos\theta_2 & 1 \\
\vdots & \vdots & \vdots \\
\sin\theta_n & \cos\theta_n & 1
\end{bmatrix} \tag{7}
$$

To promote the GDOP and other values, the geometric model for two-dimensional positioning is proposed in Fig. 3. In the figure, one can imagine that the triangles are formed using three unit vectors points toward the transmitters and the azimuth is measured in 360 degrees with clockwise.

Here the 3×3 covariance matrix would be selected for applying to two-dimensional radio navigation or positioning for the objective of this study.

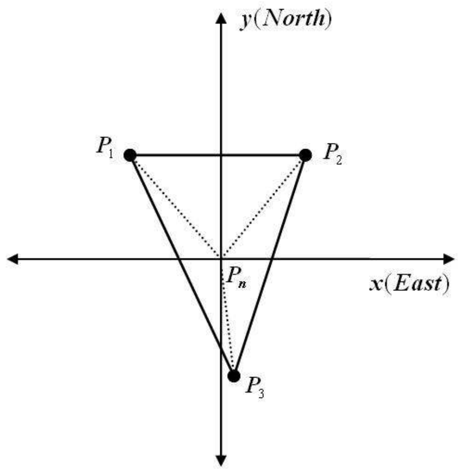

Fig. 3. The Proposed Geometric Model for Two-Dimensional Positioning

In addition, with recalling the well known the covariance matrix, the expression of the covariance matrix is represented by

$$COV(\delta X) = \begin{bmatrix} \sigma_{x_u}^2 & \sigma_{x_u y_u}^2 & \sigma_{x_u t_u}^2 \\ \sigma_{x_u y_u}^2 & \sigma_{y_u}^2 & \sigma_{y_u t_u}^2 \\ \sigma_{x_u t_u}^2 & \sigma_{y_u t_u}^2 & \sigma_{t_u}^2 \end{bmatrix} \tag{8}$$

$$= \sigma_r (G^T G)^{-1}$$

In the above formulas, $(G^T G)^{-1}$known the GDOP matrix is the matrix of multipliers of ranging variance to give position variance. And σ_r is the pseudorange error factor. Then GDOP can be mathematically computed as trace of the $(G^T G)^{-1}$matrix as follow:

$$(G^T G)^{-1} = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \tag{8}$$

$$\begin{aligned} GDOP &= \sqrt{trace_3 (G^T G)^{-1}} \\ &= \sqrt{G_{11} + G_{22} + G_{33}} \end{aligned} \tag{9}$$

In case of two-dimensional case, the positional DOP is equal to the horizontal DOP,

$$\begin{aligned} PDOP &= HDOP \\ &= \sqrt{trace_2 (G^T G)^{-1}} \\ &= \sqrt{G_{11} + G_{22}} \end{aligned} \tag{10}$$

Finally, the accuracy of two-dimensional position determined by the receiver can be expressed in terms of DOP values and the pseudorange error factor, σ_r as follow:

$$\begin{aligned} \sqrt{\sigma_{X_u}^2 + \sigma_{y_u}^2} &= PDOP \times \sigma_r \\ &= HDOP \times \sigma_r \end{aligned} \tag{11}$$

Ⅳ. DISCUSSION & CONCLUSION

We have studied on algorithms for two-dimensional positioning based on GPS Pseudorange Technique.

The main works and results are summarized below. First, the linearized state equation was mathematically derived based on GPS pseudorange technique. Second, the geometry model with respect to triangles formed using unit-vectors were proposed for investigation of land-based radio positioning. Finally, the corresponding mathematical formulations for DOP values and covariance matrix were designed for two-dimensional positioning.

In the future works, we would like to continue to apply the resultant algorithms to various practical fields such as GPS jammer, eLORAN and other land-based radio positioning systems.

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