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High School Mathematical Education of Future Physicists[1]

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Concordance of high school courses of mathematics and physics is a long-known and still-unsolved problem, at least in Russia. Lyceum “Physical-Technical High School” exists for more than 20 years and endeavors to solve this problem. During this work, Lyceum teachers worked out certain ideology of educational content as well as methods of teaching specific topics. Textbooks and workbooks have been written for the Lyceum students by the Lyceum teachers (or in collaboration with them).
This article reports on the cumulate experience of the Lyceum in mathematical education of future physicists.

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1. INTRODUCTION

This article reports on the experience of teaching mathematics in a high school that is oriented on preparation of future physicists. The school was created more than 20 years ago after the initiative of Nobel Prize winner Zhores Alfyorov. The main task of the school is to form a future researcher.

The school is interested in the research work of a student both on the lesson and outside it. In particular, school students take part in profession scientific process in

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scientific laboratories of academic institutes of St. Petersburg. Student of the school are winner of many achievements, Olympiads and contests in physics and mathematics of city, state and international levels.

School teachers are of very high scientific and methodical qualification. They have written a number of textbooks and participate in state and international conferences on a regular basis.

The school has conducted courses for students of other cities of Russia and foreign countries, *e.g.*, from South Korea.

2. DISCONNECTION OF PHYSICS AND MATH COURSES

Connection between physics and mathematics was written about by many scientists, such as Poincare (Manin, 2008), Einstein (Arnold, 2000), and Weyl (1989); Einstein even considered geometry “simply the oldest branch of physics” (Arnold, 2000). As soon as physics and mathematics are connected in their essence, it seems reasonable to connect their school courses. But it is not common. Many math teachers are convinces that mathematics is all-sufficient. It might seem that only few people think seriously about the necessity of connecting school math and school physics. On the other hand, many physicists (academicians Landau, Zeldovich & Taglom, 1982) emphasized that teaching mathematics to physicists and teaching mathematics to mathematicians should differ.

Here are some typical examples of discrepancy in explanation of the same phenomena, from the Russian experience. **Derivatives** and **integrals** are needed in the physics course much earlier than they are introduced in the math course. Different definitions of **vector** are given; and in each student’s head these two concepts must merge into one – in some mysterious way. Some mathematical concepts used in school physics course are missing in the math course at all, *e.g.*, **normal, cross product, vector function** and its **derivative** etc. As for approximate calculation, this is a pure trouble. Calculating the volume of a cube with edge length 1.25 cm would lead to different results on a math lesson and on an experimental physics laboratory (say, experimental measurement of the density of some material). On a math lesson, the answer would have all the digits after the decimal point (decimal comma in Russia), whereas giving such precise answer in the physics lab would be erroneous due to the obvious impreciseness of the initial data.

Not finding a consensus with math teachers, physics teacher dare to introduce the needed mathematical concepts themselves. In the book “Mathematics as Pedagogical Task”, Freudenthal (1977) classifies this as some kind of schizophrenia.

We think that teaching math to physicists has its own specificity, compared to teaching math to mathematicians. The talented students do not fully understand the efforts to pay lots of attention to the question of existence. For example, they ask: “Why prove the existence of the solution of harmonic oscillation equation?” They see the process and they understand the there is a solution that satisfies all the conditions.

It is clear that for a school that sets physics as a priority subject this is an abnormal situation. Mathematician (academician) Alexandrov (1988) said,

> “Connection to the physics, to the practical experience — is not only a methodical necessity, but is also representation of the essence of the science and education — because a science can only exist in connections, in interaction with other sciences, technical tasks and real life. Education must give elements of science in this sense, not formal knowledge that is learned to get better grades.”

But how can we solve this problem in the actual teaching process?

3. RATIONAL REASONING

One of the authors of this article uses the geometry textbooks written with his collaboration (Alexandrov, Verner & Ryzhik, 2009). These textbooks give rich and various presentations of physics — the center of mass, dot and cross product, special relativity theory. The definition of vector is given in the way it used in physics.

The textbooks contain even physics-related problems. They not only cover the underlying theory, but also supplement it, as with the Pappus-Guldinus theorem. There are problems concerning trajectory of a material point, the shape of the rainbow, finding the forces in a metal shank. The center of mass is used to solve the following problem: prove that for an arbitrary point inside a convex polygon, at least one its projection onto the plane of a face falls into the face.

Both in theoretical texts and in problem solutions, rational reasoning is allowed that is common in physics but not always customary in mathematics, such as:

- Reasoning that can be strictly proved in pure mathematics, but at the current point the techniques described in the textbook are not powerful enough for such proofs,
- Reasoning that is obvious from pictorial argumentation (visually).

The four examples of such reasoning are: symmetry, cinematic processes, continuity and illustrations.

3.1 Symmetry

Example. Prove that it is possible to inscribe a given regular pyramid into a sphere. Such proof can start with a phrase: “Using the symmetry of the pyramid, let’s search for the possible center of the sphere only on the height of the pyramid or its continuation.

3.2 Cinematic processes

Example. Find the largest possible value of the volume of a tetrahedron, given that four of its six edges equal 1, and three of these four edges are edges of one face.

The three edges form an equilateral triangle; call it the base of the tetrahedron. Let the fourth edge of length 1 "hang freely in the space", begin attached with its one endpoint to a vertex of the base. The height of the tetrahedron cannot be greater than the length of the hanging edge. And the equality only holds when the hanging edge is perpendicular to the base.

Of special interest is the method that uses radius-vectors and elements of calculus applied to vector functions. Here's an example that illustrates this method.

Example. Two sides of the given triangle are hypotenuses of two outside-directed right isosceles triangles. Prove that the segment connecting the vertices of their right angles is seen from the midpoint of the third side of the given triangle at a right angle.

Solution. Denote the triangle *ABC* and the two right triangles *CAN* and *BCM*. Denote the midpoint of *AB* by *D*.

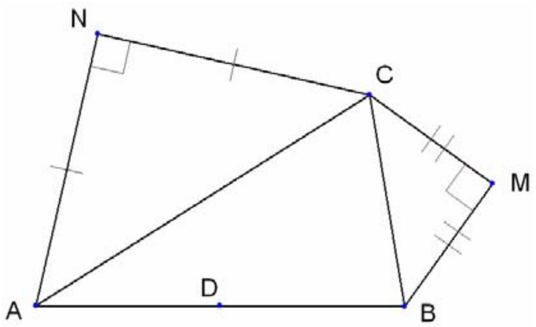

Figure 1.

Note that $\vec{AN} = \frac{1}{\sqrt{2}} \vec{AC}^{45^\circ}$ and $\vec{BM} = \frac{1}{\sqrt{2}} \vec{BC}^{-45^\circ}$.

(These notations mean that vectors

$$\vec{AC} \text{ and } \vec{BC}$$

are rotated the corresponding number of degrees counterclockwise).

Now suppose that points *A* and *B* are fixed and point *C* is moving with some velocity. Then the velocities of the vectors are connected (in any base) with the following equalities:

$$\vec{V}_N = \frac{1}{\sqrt{2}} \vec{V}_C^{\,45°} \tag{1}$$

and

$$\vec{V}_M = \frac{1}{\sqrt{2}} \vec{V}_C^{\,-45°} \tag{2}$$

From (2) we have:

$$\vec{V}_C = \sqrt{2}\ \vec{V}_M^{\,45°}$$

Substitute $\vec{V}_C$ with this expression in (1) and get:

$$\vec{V}_N = \vec{V}_M^{\,90°}$$

Therefore, by integrating this vector equality, we receive:

$$\vec{r}_N = \vec{r}_M^{\,90°} + \vec{r}_0,$$

where $\vec{r}_0$ is a constant vector. Now set the point *D* as the origin.

To find $\vec{r}_0$, consider the position of the point *C* that makes the triangle *ABC* right and isosceles with *AC*=*BC*. In this case, clearly

$$\vec{r}_N = \vec{r}_M^{\,90^0}, \text{ which gives } \vec{r}_0 = \vec{0}.$$

Thus, when *D* is the origin, for any position of point *C*: $\vec{r}_N = \vec{r}_M^{\,90^0}$, q. e. d.

3.3 Continuity

The idea of continuity is used in three types of problems.

Example (method of small perturbations). Find whether a section of a regular tetrahedron can be an obtuse triangle.

Example (continuously changing value). Consider a point X moving along certain curve from point X_1 to point X_2, and some value $V(X)$ changing continuously, such that $V(X_1) < P < V(X_2)$. Then there is a point X_0 on this line such that $V(X_0) = P$. That's how we prove the existence of an inscribed sphere for any regular pyramid, for example.

Example (proceeding to limit). If some geometrical figure or value has some property in all states of a continuous process, then this property will (in some cases) also hold in the limit state of the process. That's how we prove the test of perpendicularity of a line and a plane.

In all of these cases, the continuity is not proven but rather perceived intuitively.

3.4 Illustration

Use of illustration is acceptable if the solution of a problem (or its part) is so obvious from the picture that an attempt to prove the result formally is taken by students with bewilderment. Here's an example.

Example. A square was rotated 45° about its center. What is the ratio of the area of the square to the area of the intersection of the squares? Now, who would formally prove that the intersection in an octagon? Everyone would simply count the number of sides on the picture.

4. STRICTNESS OF FORMULATIONS

The textbook also uses other trends seen in physics. Say, the approach to the question of existence of a mathematical object is close to the approach used in physics. If we see that a prototype of a mathematical object exists in the real world (especially if can be observed), then we say that the mathematical object itself also exists.

For example, in the very beginning of the stereometry course, we use the existence of a regular tetrahedron – as it can be folded (assembled) from a net; the existence of a sphere – as there is a football.

Also we rely on existence of something that was actually constructed by student, say an isosceles triangle constructed by compass and straightedge.

This general approach to the question of existence of an object allows to do stereometry in the planimetry course. So, a perpendicular to a plane, several polyhedrons and a sphere are shown to the student in the planimetry course.

Furthermore, it is well-known that for a physicist it is important to formulate the question and the conditions of the problems in a correct way. There are problems in the textbook that demand such work to be done.

We are too used to the practice that the problem statement contains everything we need – the question is set in a precise manner, the conditions are described unambiguously, and the given data are just enough to solve the problem.

So, the textbook contains problems with too much or too little data, as well as those where the statements can be interpreted and completed rather freely according to the student's understanding of the situation.

5. EXAMPLE: DERIVATIVE

Some parts of mathematics are especially important in physics. Say, elements of calculus. And it happens that in the mechanics course the concept of derivative is required before it is studied in the mathematics course.

Can we teach the concept of the derivative before the concept of limits? We can start with a conversation about the instantaneous speed. The concept of instantaneous speed is by itself contradictory: on one hand there is no time interval – so the speed can't be defined in the usual way, on the other hand, there is some movement, so there must be some speed – something a speedometer arrows can show. The general logic of teaching here is the following: first, understand what we want from the concept of instantaneous speed; then, based on the obtained understanding, give a procedure for its calculation; and finally, turn this procedure into a constructive definition.

On this way, it is hard to understand what a material point is, what a "moment of time" is. So it is very handy to use more comprehensible images here: switch the conversation from instantaneous speed to the slope of the graph of a function – a property that is vividly clear to everyone and that has practical basis: everybody has done mountain skiing, or at least climbed a stairs.

It is also possible to speak about the slope of the tangent instead of the slope of the graph, but it is less comprehensible since we don't meet very many tangents in real life. Looking at the graph of a function $y = x^2$ with $x > 0$, it can be seen (without any formal definition) that the greater is the value of x, the "steeper" is the curve: "From what point would you dare to slide down on the skis, if a mountain looked like this curve?"

The further train of thought is the following: our general task is to turn a qualitative property into a quantitative, *i.e.*, to find a way to calculate it. And the result of the calculation must confirm what we perceive from the visual reasoning.

Solve this task for a particular case – calculate the slope of a parabola $y = x^2$ in $x = 1$. Consider the graph of $y = x^2$ with $x > 1$.

The slope of any straight line passing through point (1, 1) is known, whereas the slope of the parabola is unknown yet. Try to bind the known and the unknown, in this case, express the slope of the parabola in terms of the slope of a straight line. But there are many straight lines passing through the point (1, 1). Which one is the only one, whose slope we should chose as the slope of the parabola?

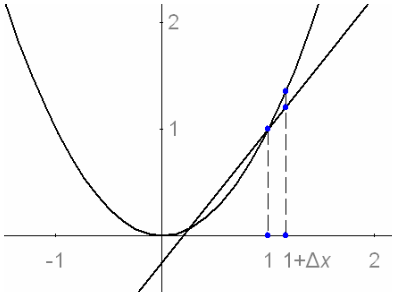

Figure 2.

We see that in $x = 1$ the parabola and the straight line coincide, whereas in $x = 1 + \Delta x$ they differ. The difference equals:

$$y_1 - y_2 = ((1 + \Delta x)^2 - 1) - k\cdot\Delta x = 2\cdot\Delta x + \Delta x^2 - k\cdot\Delta x = (2 - k)\cdot\Delta x + (\Delta x)^2$$

Looking at the formula, we see that the difference between the parabola and the line consists of two terms. The first term is proportional to Δx, and therefore it is more significant than the second, which becomes negligible at sufficiently small values of Δx (this statement can be easily illustrated numerically).

Since we want the slope of the parabola and the slope of the line to be equal, we would like to make the first term equal zero. That happens when $k = 2$. Thus, the slope of the parabola at the point $x = 1$ should be considered equal to 2.

In a similar way we solve the same task for an arbitrary point x_0, in which the slope turns out to be equal to $2x_0$.

Now it is useful to make sure that this approach to calculating slopes in “good”, that is, the result matches out intuitive understanding of steepness. Indeed, this formula is supported by the observation that the greater is x, the steeper is the parabola. (The situation for negative values of x is also worthy of conversation, but this is not essential.)

At this point it is possible to proceed to other specific curves, and after some examples – to the definition of the slope of the graph of an arbitrary function.

You can then move on to other specific curves (this approach does not always lead to the result, but at the moment is enough, when I can), and then to determine the slope for an arbitrary function.

Now it is appropriate to return to the problem of uneven velocity in a straight line. The solution is almost verbatim repeat solution of the slope.

Comparing the slope and speed of finding a non-uniform motion, we see that to solve two different problems by their nature Mathematics offers the same approach. This opens the way for formal, purely mathematical discussion of the derivative.

(In fact, the concept of slope is about the question of the best approximation of curve with a straight line. It can come out from the area of approximate calculations, which is especially important due to the use of computers. But this is too complex for the beginning of a conversation. It's better to start mathematical conversation with the students with a description of a vivid problematic situation; ideally–with a practically tangible one; even better–with actually visible one).

6. MATHEMATICAL EXPERIMENTS

Mathematician (academician) Arnold says Arnold (2000) that "mathematics is an experimental science"– as well as physics. This statement is supported by the increasing role of computers Moiseev (1979).

In the process of teaching math, computers have the same role as measurement instruments in physics. When physicists see something on the screen of their oscillographs, they either use the observed results in their future work, or try to find the explanation for the observed results. The same applies to the use of computers when teaching math.

There are many aspects. In particular, a measurement tool can produce strange results – the results produced by a computer can also be strange. It is useful to conceive the anticipated result beforehand. If the actual result follows the expectations, this serves as an evidence for the current hypothesis, otherwise, one should check both the reasonability of the hypothesis and the correctness of the experiment. For example, a student might assume that the smallest circle containing a given triangle is the outscribed circle – but a correctly set experiment would refute his erroneous assumption.

Experimental search for a mathematical pattern – is not something new at all. Poya (1965) wrote minutely about it in his books. There is a fundamental difference between a finite number of experiments carried out in practice, without a computer, and computer experiments, at least in geometry. In a computer simulation, free movement of objects on a screen results in numerous different situations – that can be seen as infinitely numerous. Therefore, the result is psychologically convincing, and we can assume that it was received as a result of **computer proof**.

Poincare wrote that a proof is a check. If we take his point of view and consider what we see on the screen as a check (of some assumption we made), then the result of computer experiment is the proof of our assumption.

Mention one more similarity between a physicist's work with an instrument and a mathematician's work with a computer. A computer allows observing the behavior of an object during some process – and see some property of an object changing dynamically. However, in physics the time is a fundamental concept, whereas in mathematics it is just an auxiliary tool.

7. CONCLUSION

It seems reasonable to believe that our approach to mathematical education of future physicists might also be helpful in teaching future engineers and students with other priorities, possibly even humanities-bound students.

To conclude we will quote a conversation that happened on the lesson.

A geometrical problem (on projection of a point inside a convex polyhedron onto its faces) was solved using physical reasoning. After that, the students were asked a question:

> "Do you object to using physics in solving mathematical problems?"

One of the answers was the following:

> "If we use math when learning physics, why can't we use physics when learning math?"

The other answer was more comprehensive:

> "Physics, as well as mathematics, does deal with abstract concepts. But the abstract concepts in mathematics are simpler than those in physics. In this sense, physics is a simpler science that mathematics. And it is natural to explain something complex using something simple, and not the other way around."

This started a fruitful discussion on which approach is more correct.

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