AN ISOLATED POINT IN A PARTLY ORDERED SET WITH INTERVAL TOPOLOGY

By Tae Ho Choe

1. Introduction.

The problem of finding necessary and sufficient condition for an element of a complete lattice to be isolated in the interval topology posed by Birkhoff [1][1]. It has already been solved in the case that for an element x to be isolated to the lattice L itself in it's interval topology by Northam [2], that is, the necessary and sufficient conditions are the followings:

(a) x covers a finite number of elements and every element under x is under an element covered by x.

(b) x is covered by a finite number of elements and every element over x is over an element which covers x.

(c) x belongs to a finite separeating set[2] of L in which no other member is comparable with x.

In this short paper, we shall find a necessary and sufficient conditions for an element of a partly ordered set P to be isolated to a subset M of P. And we shall give some Remarks which shows our conditions are equivalent to Notham's conditions (a), (b), (c) if the subset M be P.

We here recollect some standard terms. P is *partly ordered* if it is subject to a binary relation $\leqq$ which is reflexive, antisymmetric, and transitive. In P, if neither $x \leqq y$ nor $y \leqq x$, then x and y are said to be *incomparable* and this is denoted by $x \# y$. The *interval topology* for P is defined by taking as a sub-basis for the closed sets the class S of all

1) Number in brackets represent references listed at the end of the paper.

2) Northam defines a separating set for closed intervals in the following way. Let x and y be two elements in a partially ordered set, with $x<y$. A set of elements (a_i) is called a separating set for the closed interval $[x, y]$ if $x<a_i<y$ all i, and every element in $[x, y]$ is comparable with at least one a_i. This requires that intervals containig less than three elements are said to be separated by the empty set.

sets (half intervals) of the form $\{x : x \leqq a\}$ and $\{x : a \leqq x\}$. It is convenient to introduce the notation $L(a)$ and $M(a)$ to denote, respectively, the proceeding half intervals. By a *covering* of an arbitrary set A, we mean a collection of subsets of P whose union includes A. we let A' denote the complement of set A.

2. The isolated point in the interval topology

The following Lemma is obvious.

[LEMMA] *In a topological space P, if $F=\{B_\alpha : B_\alpha \subseteq P\}$ is a base of open sets then $F'=\{B'_\alpha : B_\alpha \in F\}$ is a base of closed sets.*

[THEOREM] *A necessary and sufficient condition for an element a of a partly ordered set P to be isolated to subset M in the interval topology is that for the element a there exist the finite subsets A and B such that*

(i) $A=\{x : x \# a \text{ or } x>a\}$, $B=\{y : y \# a \text{ or } y<a\}$

(ii) $(M(x))_{x \in A}$, $(L(y))_{y \in B}$ *are covering of* $M-a$

[PROOF] At first we hall show that the conditions (i), (ii) are necessary. If a is isolated to M, then $a \notin \overline{M-a}$.

And we can find a neighbourhood $V(a)$ of a that $V(a) \cap [M-a]=0$.

Since $V(a)$ is an open set, there exist basic open sets U_β such that $V(a)=\bigcup_\beta U_\beta$, where, $U_\beta \cap [M-a]=0$ for all β.

Therefore, there exist at least one basic open set U_β containing a.

Since U'_β is a basic closed set by Lemma, U'_β must be the union of a finite number of sub-basic of closed set.

Hence there are two finite subsets A, B of P such that

$$U'_\beta=[\bigcup_{x \in A} M(x)] \cup [\bigcup_{y \in B} L(y)]$$

which includes $M-a$ because $U_\beta \cap [M-a]=0$.

If $x \leqq a$ and $x \in A$, then $a \in M(x)$ which is contrary.

Hence any element x of A is either $x \# a$ or $x>a$. Similarly, we have any element y of B is either $y \# a$ or $y<a$.

we now consider the suffciency. In an interval topology, $\overline{M-a}$ is the intersection of

$$\Gamma=\bigcap_\alpha \{F_\alpha : F_\alpha \text{ is closed subset such that } M-a \subset F_\alpha \subset P\}.$$

On the other hand, the subset $[\bigcup_{x\in A} M(x)] \cup [\bigcup_{y\in B} L(y)]$ is a closed subset of P and includes $M-a$. Threfore,

$$[\bigcup_{x\in A} M(x)] \cup [\bigcup_{y\in B} L(y)] \in \Gamma$$

while $a \notin [\bigcup_{x\in A} M(x)] \cup [\bigcup_{y\in B} L(y)]$, infact, if $a \in M(x$ for some $x \in A$ then $a \geqq x$, which is contrary, and similarly if $a \in L(y)$ for some $y \in B$ then $a \leqq y$, which is also contrary.
Hence $a \notin \overline{M-a}$.

[REMARKS] In particular case of $M=P$, our conditions are equivalent to Northam's conditions (a), (b), (c). For, there exist a finite number of maximal elements of $\{x : a > x\}$ in B, and a finite number of minimal elements in A, since $(M(x))_{x\in A}$, $(L(y))_{y\in B}$ are a covering of $P-a$.

Therefore, the conditions (a) and (b) hold, and the condition (c) easily hold, too. And it is easy to see the converse if M P.

Finally, we here give as example that the conditions (a), (b) are unnecessary for element a of M to be isolated point of M $(\neq P)$. Let P be the topological space to all real numbers and M be the subset

$$\{ 0,\ x : 1 < x,\ or\ -1 > x \} \text{ of } P.$$

Then zero is an isloated point of M since $M(1) \cup L(-1)$ includes $M-0$. However, there is any element in M neither covering zero nor being covered by zero.

May 20. 1958
Department of mathematics
Liberal Art and Science Collage
Kyungpook Univercity

REFERENCES

[1] G. Birkhoff, Lattice Theory, rev. ed., New York (1948)
[2] E. S. Northam, The Interval Topology of a Lattice, Proc. Amer. Math. Soc. 4 (1953)