NOTE ON A MAPPING OF CONNECTED COMPONENTS

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Let \( f \) be one-to-one bicontinuous mapping between a topological space \( X \) and a topological space \( Y \). Then \( X \) and \( Y \) are homeomorphic, so that there exists one-to-one correspondence between connected components of \( X \) and connected components of \( Y \), by \( f \). But there is the another condition for this matter, where \( f \) is not a homeomorphic mapping.

The correspondence \( f \) is closed if \( A \) a closed subset of \( X \) then \( f(A) \) is closed in \( Y \); \( f \) is connected if \( C \) a connected subset of \( X \) implies \( f(C) \) is a connected subset of \( Y \). [2]

For mapping \( f \), if \( f^{-1} \)is closed then \( f \) is continuous mapping from \( X \) to \( Y \). [1]

**Lemma 1.** \( f \) is closed (preimage of \( f \), then \( f \) is connected. [1]

We assume the following property for mapping \( f \):

(P) For each closed subsets \( A, B \), \( f(A) \cap f(B) \neq \emptyset \)

there exists at least one element \( y \in f(A) \cap f(B) \)

such that there is some \( x \in f^{-1}(y) \) having sequences \( \{a_n\} \subseteq A, \{b_n\} \subseteq B, \; a_n \to x, \; b_n \to x \).

**Lemma 2.** \( f \) is closed mapping having property (P). If \( A, B \) are closed sets: \( A \cap B = \emptyset \), and then \( f(A) \cap f(B) = \emptyset \).

Proof. If \( f(A) \cap f(B) \neq \emptyset \), there exists a \( x \in A \cap B \), by condition (P).

Since \( A \cap B = \emptyset \) and \( A, B \) are closed sets, then \( f(A) \cap f(B) = \emptyset \).

**Lemma 3.** \( f \) is closed mapping with property (P). For any set \( M \) of \( Y \), if \( A, B \) are closed sets of \( X \) such that \( A \cap B \cap f^{-1}(M) = \emptyset \), then \( f(A), f(B) \) are disjoint in \( M \).

Proof. If it is not so: \( f(A) \cap f(B) \cap M \neq \emptyset \). There is a \( x \) belonging to \( f^{-1}(y) \), such that \( a_n \to x, \; b_n \to x, \; \{a_n\} \subseteq A, \; \{b_n\} \subseteq B \) and \( x \in f^{-1}(M) \), because

\[
  f^{-1}(y) \subseteq f^{-1}[f(A) \cap f(B) \cap M] = f^{-1}[f(A) \cap M] \cap f^{-1}[f(B) \cap M] = f^{-1}[f(A) \cap f^{-1}(M) \cap f^{-1}(M)].
\]
Hence \( x \equiv \overline{A} \cap \overline{B} \cap \overline{f(M)} = A \cap B \cap f^{-1}(M) \).

Since \( A, B \) is disjoint in \( f(M) \), then \( f(A), f(B) \) disjoint in \( M \).

LEMMA 4. \( f \) is biclosed mapping with property (P) and \( C \) is a connected component of \( X \), then \( f(C) \) is connected component of \( Y \).

Proof. \( f(C) \) is connected by lemma 1. For any connected subset \( C' \) of \( Y \) containing \( f(C) \), \( f^{-1}(C') \) contains \( C \).

Now \( A, B \) are closed sets of \( X \) such that \( f^{-1}(C') \subset A \cup B \), and \( A, B \) are disjoint in \( f^{-1}(C') \), then by using lemma 3, \( f(A), f(B) \) are disjoint in \( C' \). \( f(A), f(B) \) are closed, for \( f \) is closed mapping, and \( f(A) \cup f(B) \supset C' \). Since \( C' \) is connected set in \( Y \), \( f(A) \cap C' = \emptyset \) or \( f(B) \cap C' = \emptyset \). Then \( A \cap f^{-1}(C') = \emptyset \), or \( B \cap f^{-1}(C') = \emptyset \), for \( A \cap f^{-1}(C') \subset f^{-1}(f(A) \cap f^{-1}(C')) = f^{-1}(f(A) \cap C') = \emptyset \).

Hence \( f^{-1}(C') \) is connected. Then \( f^{-1}(C') = C \), because \( C \) is a connected component of \( X \) and \( C \) is contained in the connected set \( f^{-1}(C') \). So that \( C' \) is equal to \( f(C) \), i.e. \( f(C) \) is a connected component of \( Y \).

REMARK. \( f \) is closed mapping with property (P), then if \( M \) disconnected in \( X \) implies \( f(M) \) is a disconnected set of \( Y \).

THEOREM. For two topological spaces \( X, Y \), if there exists biclosed mapping with property (P), then there is one-to-one correspondence between connected components of \( X \) and connected components of \( Y \).

Proof. Let us put \( X = \bigcup_{\alpha \in I} C_{\alpha} \), \( Y = \bigcup_{\beta \in J} C'_{\beta} \), where \( C_{\alpha} \) is a connected component of \( X \) for all \( \alpha \) belonging to \( I \), and \( C'_{\beta} \) is a connected component of \( Y \) for all \( \beta \) belonging to \( J \). For each \( \alpha \) belonging to \( I \) there is a \( \beta \) belonging to \( J \), such that \( f(C_{\alpha}) = C'_{\beta} \) by lemma 4. Referring the above remark, \( f^{-1}(C'_{\beta}) \) is connected. And for the connected set \( C \) containing \( f^{-1}(C'_{\beta}) \), \( f(C) \) is contained in \( C'_{\beta} \), since \( C'_{\beta} \) is a connected component. Then \( f^{-1}(C'_{\beta}) \) is a connected component of \( X \). So there is an \( \alpha \) belonging to \( I \) such that \( f^{-1}(C'_{\beta}) = C_{\alpha} \) for each \( \beta \) of \( J \). Finally, if \( C_{1}, C_{2} \) are two connected components of \( X \), then \( C_{1}, C_{2} \) are closed and disjoint in \( X \). By lemma 2, \( f(C_{1}), f(C_{2}) \) are disjoint.
component of $Y$. Hence there is one-to-one correspondence between the elements of $I$ and the elements of $J$.

In this note $f$ is continuous closed mapping and $f$ is not one-to-one mapping.

May. 30, 1958
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REFERENCES