ON CONHARMONIC TRANSFORMATIONS IN
HERMITIAN AND KAHLERIAN SPACES

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Introduction.

The present author wishes to study a special conformal transformation which we shall call conharmonic in a Hermitian space $H_n$ and a Kaehlerian space $K_n$. The conharmonic transformations in the Riemannian spaces were already studied by Ishii [1].

The purpose of the present paper is to find a tensor whose invariability under a special conformal transformation is a necessary and sufficient condition that $H_n$ and $K_n$ are conharmonic. (Theorem 4)

1. Conharmonic transformations.

For the Hermitian space $H_n$, we denote the metric tensor, Christoffel symbols, curvature tensor by the notations $G_{{ij}}, E_{{jk}}, H_{{ijkl}}$ respectively, and for the Kaehlerian space $K_n$, referred to complex analytic coordinates $z^i = (z^a, \bar{z}^\alpha)$ ($\bar{z}^a = z^\bar{a}$), we denote by notations $g_{{ij}}, \Gamma^{\bar{a}}_{{jk}}, R^i_{{jkl}}$ respectively, then $g_{{\alpha\beta}} = g_{{\bar{\alpha}\bar{\beta}}} = 0, G_{{\alpha\beta}} = G_{{\bar{\alpha}\bar{\beta}}} = 0$ and $g_{{ij}}, G_{{ij}}$ are the symmetric and self-adjoint tensors.

We shall consider a special conformal transformation

(1.1) $G_{{\alpha\beta}} = \phi^2 g_{{\alpha\beta}}$

where $\phi(z, \bar{z})$ is a real valued function in connection with

$$
G_{{\alpha\beta}} = \frac{\partial^2 \phi}{\partial z^\alpha \partial \bar{z}^\beta} = \partial_\alpha \partial_\beta \phi
$$

Let $A$ be a harmonic function which we shall define by

1) Numbers in brackets refer to the references at the end of the paper.
2) In the following the Latin indices $i, j, k, \cdots$ are supposed to run over the range 1, 2, ..., $n$, $\bar{1}, \bar{2}, \cdots, \bar{n}$ and the Greek indices $\alpha, \beta, \gamma, \cdots$ take the values 1, 2, ..., $n$ and consequently $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \cdots$ run over the range of the symbols $\bar{1}, \bar{2}, \cdots, \bar{n}$. 
(1.2) \[ g^\alpha\beta A, \alpha\beta = 0 \]

and let seek the condition upon \( \phi \) in order that the function defined by

(1.3) \[ B = \phi^{\omega^m} A \]

may become a harmonic function with respect to the tensor \( G_{\alpha\beta} \), i.e.

\[ G^\alpha\beta B, \alpha\beta = 0 \]

where "\( \cdot \)" and "\( \cdot \)\( \cdot \)" denote covariant differentiation with respect to the tensor \( g_{\alpha\beta} \) and \( G_{\alpha\beta} \) respectively, and \( m \) is a suitable constant.

By the relations

\[ E_\alpha^\gamma = \frac{1}{2} G_{\gamma \delta} (\partial_\delta G_{\alpha \beta} + \partial_\beta G_{\gamma \delta}) \quad \text{(conj.)} \]

\[ E_\beta^\gamma = \frac{1}{2} G_{\gamma \delta} (\partial_\delta G_{\alpha \beta} - \partial_\beta G_{\gamma \delta}) \quad \text{(conj.)} \]

and Kählerian condition, we find the following relations

(1.4) \[ E_\alpha^\gamma = \Gamma^\gamma_{\alpha \delta} + \delta^\gamma_{\alpha} (\log \phi), \gamma + \delta^\gamma_{\alpha} (\log \phi), \beta \quad \text{(conj.)} \]

(1.5) \[ E_\beta^\gamma = \delta^\gamma_{\alpha} (\log \phi), \gamma - g^{\alpha \beta} g_{\beta \gamma} (\log \phi), \alpha \quad \text{(conj.)} \]

therefore

(1.6) \[ g^{\alpha \beta} B, \alpha \beta = G^{\alpha \beta} [\partial_\beta \partial_\alpha B - (\partial_\beta B) E_\alpha^\rho - (\partial_\alpha B) E_\beta^\rho] = 2m (2m - 3 + 2n) \delta^{(n-3)} g^{\rho \beta} \phi_\rho \phi_\beta + 2mn \phi^{(n-3)} A + (2m + n - 1) \delta^{(n-1)} g^{\alpha \beta} (\phi_\alpha A, \beta + \phi_\beta A, \alpha) + \phi^{(n-1)} g^{\alpha \beta} A, \alpha \beta = 0 \]

where

\[ \phi_\alpha = \phi, \alpha = \partial_\alpha \phi \quad \text{(conj.)} \]

If we determine \( m \) by

(1.7) \[ m = \frac{1 - n}{2} \]

we have from (1.2)

\[ \phi^{-%(n+1)} [(n - 2) g^{\rho \beta} \phi_\rho \phi_\beta + n\phi] A = 0 \]

Hence the required condition upon \( \phi \) is

(1.8) \[ (n - 2) g^{\rho \beta} \phi_\rho \phi_\beta + n\phi = 0 \]

We shall call the conformal transformation (1.1) satisfying (1.8) conharmonic transformation \([1]\).

Now, let us consider a vector \( A_\alpha \) of \( K_n \) and suppose that by a conformal transformation (1.1) \( A_\alpha \) be transformed into \( B_\alpha \) defined by
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(1. 9) \( B_\alpha = A_\alpha - (\log \phi), \alpha \)
then

\[
G^{\alpha \beta} B_{\alpha \beta} = \phi^{-1} \left[ g^{\alpha \beta} A_{\alpha \beta} - n \phi^{-1} + (n - 1) \phi^{-1} g^{\alpha \beta} (\phi A_{\beta} + A_\alpha \phi_{\beta}) \right] \\
+ (3 - 2n) \phi^{-2} g^{\alpha \beta} \phi_{\alpha \beta}
\]

\[
G^{\alpha \beta} B_\alpha B_\beta = (n - 1) \phi^{-2} \ g^{\alpha \beta} [A_{\alpha \beta} - \phi^{-1} (\phi A_{\beta} + A_\alpha \phi_{\beta})] \\
+ \phi^{-2} g^{\alpha \beta} \phi_{\alpha \beta}
\]

therefore

\[
G^{\alpha \beta} [B_{\alpha \beta} + (n - 1) B_\alpha B_\beta] = \phi^{-2} g^{\alpha \beta} [A_{\alpha \beta} + (n - 1) A_\alpha A_\beta] \\
- \phi^{-4} [(n - 2) g^{\alpha \beta} \phi_{\alpha \beta} + n \phi]
\]

thus we have the following theorem which is similar to Lemma in pp. 77 of [1].

THEOREM 1. A necessary and sufficient condition that the conformal transformation (1. 1) be conharmonic is that a vector field \( A_\alpha \) of \( K_n \) is transformed into a vector field \( B_\alpha \) of \( H_n \) defined by (1. 9) and satisfies the condition

\[
G^{\alpha \beta} [B_{\alpha \beta} + (n - 1) B_\alpha B_\beta] = \phi^{-2} g^{\alpha \beta} [A_{\alpha \beta} + (n - 1) A_\alpha A_\beta]
\]

2. Conharmonic invariant tensors.

By the transformation (1. 1) the non-zero components of curvature tensor of \( H_n \) are represented by the following forms

(2. 1) \( H^\alpha_{\beta\gamma\delta} = \phi^{-1} \delta^\alpha_\delta (\phi_{\beta\gamma}, - 2\phi^{-1} \phi_{\beta} \phi_{\gamma}) \)
\( - \phi^{-1} \delta^\alpha_\gamma (\phi_{\beta\gamma}, - 2\phi^{-1} \phi_{\beta} \phi_{\gamma}) \) (conj.)

(2. 2) \( H^\alpha_{\beta\gamma\delta} = R^\alpha_{\beta\gamma\delta} - 2\phi^{-1} \delta^\alpha_\delta (g_{\beta\gamma} - \phi^{-1} \phi_{\beta} \phi_{\gamma}) \)
\( + 2\phi^{-3} g_{\beta\gamma} (g^{\alpha\beta} \phi_{\delta} \phi_{\phi} - \delta^\alpha_\delta g^{\alpha\beta} \phi_{\phi} \phi_{\phi}) \) (conj.)

(2. 3) \( H^\alpha_{\beta\gamma\delta} = 2\phi^{-1} \delta^\alpha_\delta [g_{\beta\gamma} - \phi^{-1} \phi_{\beta} \phi_{\gamma} + \phi^{-1} g_{\beta\gamma} g^{\alpha\beta} \phi_{\phi} \phi_{\phi}] \)
\( - 2\phi^{-1} \delta^\alpha_\gamma [g_{\beta\gamma} - \phi^{-1} \phi_{\beta} \phi_{\gamma} + \phi^{-1} g_{\beta\gamma} g^{\alpha\beta} \phi_{\phi} \phi_{\phi}] \)
\( + 2\phi^{-2} g^{\alpha\beta} [g_{\beta\gamma} \phi_{\phi} - g_{\beta\gamma} \phi_{\phi}] \phi_{\phi} \) (conj.)

(2. 4) \( H^\alpha_{\beta\gamma\delta} = \phi^{-1} \delta^\alpha_\gamma (\phi_{\beta\delta}, - 2\phi^{-1} \phi_{\beta} \phi_{\delta}) \)
\( - \phi^{-1} g_{\beta\delta} [g^{\alpha\beta} \phi_{\phi} + g^{\alpha\beta} (\phi_{\beta} \phi_{\phi})] \)
\( - 2\phi^{-1} \phi_{\beta} \phi_{\phi} \) (conj.)
Since we have (2.6) 
\[ H^{\alpha} = \phi^{-1} g_{\alpha \beta} \left[ (\phi g^{\alpha \beta}) \phi_\beta + g^{\alpha \beta} (\phi_\beta \phi_\alpha) \right] - 2\phi^{-1} g^{\alpha \beta} \phi_\beta \phi_\alpha - 2\phi^{-1} g^{\alpha \beta} \phi_\beta \phi_\alpha + \phi_\alpha \phi_\beta \]

we have
\[ H^{\alpha} = \phi^{-2} \left( R_{\alpha \beta} + 2(1-n)\phi^{-1} g_{\alpha \beta} + 4(n-1)\phi^{-2} \phi_\alpha \phi_\beta \right) + 2(3-2n)\phi^{-2} g_{\alpha \beta} \phi_\alpha \phi_\beta \]

and if transformation (1.1) be conharmonic, then by (1.8)
\[ (2.7) \quad H^{\alpha} = R_{\alpha \beta} + 4(1-n)\phi^{-2} \left[ \phi - \frac{1}{2-n} g_{\alpha \beta} \right] \]

and on multiplying this by \( G^{\alpha \beta} = \phi^{-2} g^{\alpha \beta} \) and contracting, we get
\[ (2.8) \quad H = \phi^{-2} R \quad \text{where} \quad R = 2g^{\alpha \beta} R_{\alpha \beta}, \quad H = 2G^{\alpha \beta} H_{\alpha \beta} \]

Conversely if (2.8) hold, then from (2.6) we can see that (1.8) is satisfied, thus we have the following [1]

**THEOREM 2.** A necessary and sufficient condition that a conformal transformation (1.1) be conharmonic is that it satisfies the condition (2.8).

If the Ricci tensor is invariant under the conharmonic transformation (1.1), i.e. \( H_{\alpha \beta} = R_{\alpha \beta} \) then from (2.7) we have
\[ \phi g_{\alpha \beta} + (n-2) \phi_\beta \phi_\beta = 0 \]

but by the relation
\[ \phi (g_{\alpha \beta}) = (1-n) \left[ (\phi_\alpha \phi_\beta) \phi_\gamma + \phi_\alpha \phi_{\beta \gamma} \right] - g_{\alpha \beta} \phi_\gamma \]

and Kaehlerian condition, we have \( \phi_\alpha = 0 \) (conj.): thus we have the following
THEOREM 3. There exists no such conharmonic transformation (1. 1) that the Ricci tensor is invariant.

If transformation (1. 1) be conharmonic, then from (1. 8)

\[ H^{\alpha}_{\beta \gamma} = R^{\alpha}_{\beta \gamma} + \frac{4}{n-2} \phi^{-1} \delta^{\gamma}_{\beta} g_{\alpha \gamma} \]

and from (2. 7)

\[ \frac{1}{2(n-1)} \delta^{\gamma}_{\beta} H_{\alpha \gamma} = \frac{1}{2(n-1)} \delta^{\gamma}_{\beta} R_{\alpha \gamma} \]

\[ -2 \phi^{-2} \delta^{\gamma}_{\beta} \left[ \frac{-\phi}{2-n} g_{\alpha \gamma} - \phi \phi_{\alpha} \phi_{\gamma} \right] \]

\[ \frac{1}{2(n-1)} G_{\alpha \gamma} G^{\alpha \beta} H_{\beta \gamma} = \frac{1}{2(n-1)} \delta^{\alpha \beta} g_{\alpha \gamma} \delta^{\gamma \beta} R_{\alpha \beta} \]

\[ -2 \phi^{-1} g_{\alpha \gamma} G^{\alpha \beta} \left[ \frac{-\phi}{2-n} g_{\beta \gamma} - \phi \phi_{\beta} \phi_{\gamma} \right] \]

Hence we have

\[ H^{\alpha}_{\beta \gamma} = \frac{1}{2(n-1)} \left[ \delta^{\gamma}_{\beta} H_{\alpha \gamma} + G_{\alpha \gamma} G^{\alpha \beta} H_{\beta \gamma} \right] \]

\[ = R^{\alpha}_{\beta \gamma} + \frac{1}{2(n-1)} \left[ \delta^{\gamma}_{\beta} R_{\alpha \gamma} + G_{\alpha \gamma} G^{\alpha \beta} R_{\beta \gamma} \right] \]

therefore

\[ C^{\alpha}_{\beta \gamma} = R^{\alpha}_{\beta \gamma} - \frac{1}{2(n-1)} \left[ \delta^{\gamma}_{\beta} R_{\alpha \gamma} + G_{\alpha \gamma} G^{\alpha \beta} R_{\beta \gamma} \right] \]

is invariant under the conharmonic transformation (1. 1).

Conversely if \( C^{\alpha}_{\beta \gamma} \) be invariant under the transformation (1. 1), we have from (2. 6)

\[ H^{\alpha}_{\beta \gamma} - R^{\alpha}_{\beta \gamma} = 2 \phi^{-2} \left[ \delta^{\gamma}_{\beta} \phi_{\alpha} \phi_{\gamma} + \phi \phi_{\beta} \phi_{\alpha} \phi_{\gamma} \right] \]

\[ + \frac{2}{n-1} \phi^{-1} \delta^{\gamma}_{\beta} g_{\alpha \gamma} [(1-2n) + (3-2n) \phi^{-1} g^{\beta \gamma} \phi_{\beta} \phi_{\gamma}] \]

comparing this relation with the relation (2. 2) we have

\[ \frac{2}{n-1} \phi^{-1} \delta^{\gamma}_{\beta} g_{\alpha \gamma} [(n + (n-2) \phi^{-1} g^{\beta \gamma} \phi_{\beta} \phi_{\gamma}] = 0 \]

this relation is equivalent to (1. 8), thus we have the following
Theorem 4. A necessary and sufficient condition that the conformal transformation (1.1) be conharmonic is that the tensor $C^\alpha{}_{\beta\gamma\delta}$ is invariant under the transformation (1.1).

If $C^\alpha{}_{\beta\gamma\delta}=0$, from (2.11) we get by contraction

$$R_{\alpha\gamma} = \frac{R}{2(n-2)} g_{\alpha\gamma} = 0$$

and on multiplying this by $g_{\alpha\gamma}$ and contracting, we get $R=0$, and by Theorem 2 we have the following

Theorem 5. If the transformation (1.1) be conharmonic and $C^\alpha{}_{\beta\gamma\delta}$ be a zero tensor in $K_\eta$, then $R=0$ and also $H=0$ in $H_\eta$.

3. Conharmonic sectional curvatures

If the sectional curvature $L$ of $H_\eta$ be same for all possible 2-dimensional section, then the curvature tensor must have the following form

$$H_{\kappa\nu\lambda} = L(G_\kappa G_{\nu\lambda} - G_{\nu\lambda} G_\kappa)$$

but in the present case this reduce to

$$(3.1) \quad H_{\beta\gamma\tau}=L(G_{\gamma\tau} G_{\beta\tau} - G_{\beta\tau} G_{\gamma\tau}) \quad \text{(conj.)}$$

$$H_{\alpha\beta\tau}=LG_{\alpha\beta} G_{\gamma\tau} \quad \text{(conj.)}$$

and on substituting this into $H_{\alpha\beta\gamma}$, we find

$$(3.2) \quad H_{\beta\gamma}=H^\beta_{\beta\gamma} + H^\beta_{\beta\gamma} = 2(n-1) L G_{\beta\gamma}$$

If $K_\eta$ be a space of constant holomorphic curvature, we have (pp. 162 of [2])

$$(3.3) \quad R_{\alpha\beta\gamma\delta} - \frac{1}{2(n+1)} (g_{\alpha\beta} R_{\gamma\delta} + g_{\alpha\gamma} R_{\beta\delta} + g_{\beta\gamma} R_{\alpha\delta} + g_{\beta\delta} R_{\alpha\gamma}) = 0$$

$$(3.4) \quad R_{\alpha\beta\gamma} = \frac{R}{2n} g_{\alpha\beta\gamma} = \frac{R}{2n} \phi^{-2} G_{\alpha\beta\gamma}$$

and by (2.11) we have

$$(3.5) \quad g_{\alpha\beta} R_{\gamma\delta} + g_{\gamma\delta} R_{\alpha\beta} = 2(n-1) (R_{\alpha\beta\gamma\delta} - C_{\alpha\beta\gamma\delta})$$

and on substituting this into (3.3) and using the relations
we have
(3.6) \( (n-3) R_{\alpha \beta \gamma \delta} = 2(n-1) C_{\alpha \beta \gamma \delta} \)

Next, \( C_{\alpha \beta \gamma \delta} \) is invariant under the conharmonic transformation (1.1); therefore from (2.10)
(3.7) \( C_{\alpha \beta \gamma \delta} = \phi^{-2} [H_{\alpha \beta \gamma \delta} - \frac{1}{2(n-1)} (G_{\alpha \beta} H_{\gamma \delta} + G_{\gamma \delta} H_{\alpha \beta})] \)
and on substituting (3.4) into (2.10)
(3.8) \( R_{\alpha \beta \gamma \delta} = \phi^{-2} [H_{\alpha \beta \gamma \delta} - \frac{1}{2(n-1)} (G_{\alpha \beta} H_{\gamma \delta} + G_{\gamma \delta} H_{\alpha \beta}) + \frac{R \phi^{-2}}{2n(n-1)} G_{\alpha \beta} G_{\gamma \delta}] \)

By substituting (3.7) and (3.8) into (3.6), and using (3.1), (3.2) and (2.8) we find
(3.9) \( L = \frac{1}{n(n+1)} H \)
therefore we obtain the following conclusion.

THEOREM 6. If \( H_n \) whose sectional curvature \( L \) is same for all possible 2-dimensional sections is transformed, by the conharmonic transformation (1.1), from \( K_n \) which is a space of constant holomorphic curvature, then we have the relation (3.9).

If we assume that at all points of the space \( H_n \), the holomorphic sectional curvature is all the same, then we must have
(3.11) \( H_{\alpha \beta \gamma \delta} = L (G_{\beta \gamma} G_{\alpha \delta} - G_{\beta \delta} G_{\alpha \gamma}) \) (conj.)
(3.12) \( H_{\alpha \beta \gamma \delta} = - L G_{\alpha \delta} G_{\gamma \beta} \) (conj.)
(3.13) \( H_{\alpha \beta \gamma \delta} = H_{\alpha \beta \gamma \delta} = 0 \) (conj.)

If (1.1) be conharmonic transformation, then by (2.1), (2.2), (2.3), (2.5) and (1.8) we obtain
From (3.13) and (3.16) (or (3.17)) we get by contraction
\[ \phi_\alpha, \tau - 2 \phi^{-1} \phi_\alpha \phi_\tau = 0 \] (conj.)

and, on substituting (3.12) into (3.15), and (3.11) into (3.14), and on multiplying by \( G^\alpha G^\beta \) and \( G^\alpha G^\beta \) respectively and contracting, we get by (1. 8)
\[ L = - \phi^{-1} R \frac{2}{2n^2} \quad \text{and} \quad n(n-1) L = 0 \]

hence we conclude the following

**THEOREM 7.** There exists no Hermitian space \( H_n \) which is transformed by the conharmonic transformation (1. 1) from Kaehlerian space \( K_n \), and whose holomorphic sectional curvature is all the same and not zero at all points.
REFERENCES

