ON THE REGULARITY OF THE LOCALLY UNIFORMLY CONVEX BANACH SPACE

By Mi-Soo Bae.

1. Introduction.

A. R. Lovaglia is concerned about the locally uniformly convex Banach space connected with differentiability of the norm, [1]. In his paper Lovaglia did not concern himself with the regularity of the locally uniformly convex Banach space. It is wellknown, as shown by Kakutani [2], that the uniformly convex Banach space is regular. The uniformly convex Banach space is locally unifomly convex, but the converse theorem is not true. So we are concerned about the ploblem of whether the locally uniformly convex Banach space is regular or not. If the locally uniformly convex Banach space satisfies the condition (A), it is regular. Obviously this space is not uniformly convex in general.

2. Main theorem.

Hereafter in this paper let E be a Banach space. The Banach space E is uniformly convex if, and only if, given $\varepsilon>0$, there exists $\delta(\varepsilon)>0$, such that

$$\frac{\|x+y\|}{2}\leqq 1-\delta(\varepsilon) \text{ whenever } \|x-y\|\geqq\varepsilon, \text{ and } \|x\|=\|y\|=1.$$

E is locally uniformly convex if, and only if, given $\varepsilon>o$ and an element x with $\|x\|=1$, there exists $\delta(\varepsilon, x)>o$ and such that

$$\frac{\|x+y\|}{2}\leqq 1-\delta(\varepsilon, x) \quad \text{whenever } \|x-y\|\geqq\varepsilon \text{ and } \|y\|=1.$$

The condition (A) is that if the countable set $\{x_n\}$ of points of the locally uniformly convex Banach space E satisfies $\|x_n-x_m\|\geqq\varepsilon$ for each x_n, x_m belonging to $\{x_n\}$, and if for arbirary positive ε there exists a positive $\delta_0(\varepsilon)\leqq g.l.b.\,\delta(\varepsilon, x_n)$, where $\delta(\varepsilon, x_n)$ is some positive real number satisfying

$$\frac{\|x_n+x_m\|}{2} \leqq 1-\delta(\varepsilon, x_\circ), \text{ if } \|x_n\|=1, \quad \|x_m\|=1.$$

We need several known theorems as lemmas.

LEMMA 1. *Let E be a normed Banach space. For each f_0 of $\overline{E}$ which is the conjugate space of E, and for a given arbitrary positive real number ε there exists an x_o of E satisfying*

$$f_o(x_o) > \|f_o\| - \varepsilon, \qquad \|x_o\|=1.$$

LEMMA 2. *(Corollary of Helly's theorem) Let E be a normed linear space. $X_o(f)$ is an arbitrary bounded linear functional on $\overline{E}$. For arbitrary $f_1, f_2, \cdots, f_n$ of $\overline{E}$ and an arbitray positive real number ε there exists an element x of E satisfying $f_i(x)=X_o(f_i)$ $(i=1, 2, \cdots, n)$, $\|x\| < \|X_o\| + \varepsilon$.*

THEOREM. *The locally uniformly convex Banach space satisfying the condition (A) is a regular Banach space.*

PROOF. Let E be the locally uniformly convex Banach space satisfying the condition (A), and $X_\circ$ be an arbitrary bounded linear functional defined on $\overline{E}$.

If $X_o \equiv 0$ over $\overline{E}$, i.e. $X_o(f)=0$ for the all f of $\overline{E}$ then there is the corresponding element 0 of E such that $X(f)=f(0)=0$ for all $f \in \overline{E}$. So let us suppose that $X_o \neq 0$, then $\|X_o\| > 0$. We can suppose that $\|X_o\|=1$ without loss of generality. By lemma 1, for each $\frac{1}{n} > 0$, where n is a natural number, there exists f_n belonging to $\overline{E}$ such that $\|f_n\|=1$,

$$X_o(f_n) > \|X_o\| - \frac{1}{n} = 1 - \frac{1}{n}.$$

And we have obtained the sequence $\{f_n\}_{n=1,2,\ldots}$ of bounded linear functionals on E. For the first n elements $f_1, f_2, \cdots, f_n$ of $\{f_n\}$, and to each m $(m=1, 2, \cdots\cdots)$ there are $x_m \in E$ such that $f_i(x_m)=X_o(f_i)$,

$$\|x_m\| < \|X_o\| + \frac{1}{m} = 1 + \frac{1}{m} \quad (m=1, 2, \cdots\cdots)$$

Therefore $\lim_{n\to\infty} \|x_n\| = 1$, because

$$1-\frac{1}{n}<X_o(f_n)=f_n(x_n)\leqq \|f_n\|\cdot\|x_n\|$$

$$=\|x_n\|<1+\frac{1}{n}.$$

Now $\{x_n\}$ is a sequence of E, and we can show that $\{x_n\}$ is a strongly convergent sequence. If there exists an $\varepsilon>o$ for the infinitely many points of E, $x_{n_1}, x_{n_2}, \cdots, x_{m_1}, x_{m_2}, \cdots$ $(n_1<m_1<n_2<m_2<\cdots<n_k<m_k<\cdots)$, then $\|x_{n_k}-x_{m_k}\|\geqq\varepsilon$ $(k=1,2\cdots,)$. Since E is the locally uniformly convex there is a $\delta_k=\delta(\varepsilon, x_{n_k})>0$ such that

$$\frac{\|x_{n_k}+x_{m_k}\|}{2}<1-\delta_k \quad \text{for each } x_{n_k} \text{ and } x_{m_k} \ (n_k<m_k).$$

By the condition (A), there exists a $\delta_o\leqq g.l.b,\ \delta_k$, but with $\delta_o>0$. Hence

$$\frac{\|x_{n_k}+x_{m_k}\|}{2}<1-\delta_o \quad (k=1,\ 2,\ \cdots\cdots),$$

and therefore

$$\overline{\lim_{k\to\infty}}\ \|x_{n_k}+x_{m_k}\|\leqq 2(1-\delta_o)<2.$$

Since

$$f_{n_k}(x_{n_k})=f_{n_k}(x_{m_k})=X_o(f_{n_k}) \ for\ x_{n_k},\ x_{m_k}\ (n_k<m_k),$$

$$2(1-\frac{1}{n_k})<2X_o(f_{n_k})=f_{n_k}(x_{m_k})+f_{n_k}(x_{n_k})$$

$$=f_{n_k}(x_{n_k}+x_{m_k})\leqq\|f_{n_k}\|\cdot\|x_{n_k}+x_{m_k}\|=\|x_{n_k}+x_{m_k}\|,$$

$$\underline{\lim_{k\to\infty}}\|x_{n_k}+x_{m_k}\|\geqq\lim_{k\to\infty}2(1-\frac{1}{n_k})=2.$$

This is in contradiction with

$$\underline{\lim_{k\to\infty}}\ \|x_{n_k}+x_{m_k}\|\leqq\overline{\lim_{k\to\infty}}\ \|x_{n_k}+x_{m_k}\|.$$

Then $\{x_n\}$ must be a strongly convergent sequence, and there exists an element x_o of E such that $\lim_{k\to\infty}\|x_n-x_o\|=0$, $\|x_o\|=1$, and

$$f_n(x_o)=X_o(f_n)\ (n=1,\ 2,\ \cdots).$$

We can easily verify from the locally uniform convexity that x_o is uniquely determined by the above sequence $\{f_n\}$. If there is another x'_o of E distinct from x_o which satisfies $f_n(x'_o)=X_o(f_n)$ $(n=1,\ 2,\ \cdots)$, then $\|x_o-x'_o\|\neq0$, so that there is an $\varepsilon>0$: $\|x_o-x'_o\|=\varepsilon$, and $\|x_o+x'_o\|\leqq2[1-\delta(\varepsilon,\ x_o)]<2$ because of the locally uniform convexity. Therefore

$$2(1-\frac{1}{n}) < 2X_o(f_n)=f_n(x_o+x'_o)\leqq ||f_n|| \cdot ||x_o+x'_o|| = ||x_o+x'_o|| < 2,$$

and

$$2=\lim_{n\to\infty} 2(1-\frac{1}{n})\leqq ||x_o+x'_o|| < 2.$$

This is a contradiction.

It remains to be proved that $f_o(x_o)=X_o(f_o)$ for arbitrary $f_o \in \overline{E}$.

Let f_o be an arbitrary linear bounded functional on E. f_o and the sequence $\{f_n\}_{n=1,2,\ldots}$ with which we are concerned above make a sequence $\{f_i\}_{i=0,1,2,\ldots}$ By the same process, we can obtain the sequence $\{x'_n\}$ of points, instead of $\{x_n\}$ in the above case. $\{x'_n\}$ satisfies

$$||x'_n|| < 1+\frac{1}{n}, \quad f_i(x'_n) = X_o(f_i) \quad (i=0,1,2, \cdots; \ n=1,2,\cdots)$$

And $\{x'_n\}$ converges strongly to x'_o, where $||x'_o||=1$, and $f_i(x'_o)=X_o(f_i)$ $(i=0,1,2,\cdots)$. By the uniqueness of x_o for $\{f_n\}_{n=1,2,\ldots}$, we conclude that $x_o=x'_o$. So that $f_o(x_o)=X_o(f_o)$ for arbitrary $f_o \in \overline{E}$.

Oct 20, 1959

Mathematical Department
Liberal Arts and Science College
Kyungpook University

REFERENCES

[1] A. R. Lovaglia, *Locally uniformly convex Banach spaces*, Trans. A.M.S., Vol. 78, pp. 225—238.

[2] S. Kakutani, *The regularity of the Banach spaces*, Journal of the Physico-mathematical Society of Japan (in Japanese), Vol. 14, pp. 36—51.