LATTICE ORDERED COMMUTATIVE GROUPS OF THE SECOND KIND

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Introduction. By a partially (or lattice) ordered commutative group (po. c. g. or l. o. c. g.) we mean a set $G$ endowed with a binary operation $\cdot$ and a binary relation $\leq$ such that the following axioms are satisfied
(i) $G$ is a commutative group with respect to $\cdot$.
(ii) $G$ is a partially ordered (or lattice) by $\leq$.
(iii) if $a$ and $b$ are elements of $G$ such that $a \leq b$, then $ac \leq bc$ for all $c$ in $G$.

A. H. Clifford [1] has defined the concepts of conserver element and inverter element: in a totally ordered commutative semigroup, an element $c$ is called a conserver if $a < b$ implies $ca \leq cb$, and an inverter if $a < b$ implies $ca \geq cb$. In this note, we define similar concepts as following. Let $G$ satisfy just (i) and (ii) above. In $G$, an element $c$ is a conserver if $a \leq b$, $a \not< b$ (means $a$ and $b$ are incomparable) implies $ca \leq cb$, $ca \not> cb$, respectively, an inverter if $a < b$ (means $a \leq b$ but $a \not< b$), $a \not< b$ implies $ca \geq cb$, $ca \not< cb$, respectively. $G$ will be called a partially (or lattice) ordered commutative group of the second kind (= po. c. g. II or l. o. c. g. II) if $G$ satisfies (i), (ii) and
(iv) Every element of $G$ is either a conserver or an inverter or not both.

We call an element $d$ a destroyer if $d$ is neither conserver nor inverter, i.e., if $d$ is a destroyer, then there exists a pair of elements $x$ and $y$ in $G$ such that $x < y$ and $dx \not< dy$. We will give some typical examples of a destroyer in § 4. And we shall call po. c. g. II $G$ simply ordered commutative group of the second kind (s. o. c. g. II) if every element of $G$ is either a conserver or an inverter, and $G$ simply ordered.

Let $G_i$ ($i=1, 2, \ldots, n$) be a s. o. c. g. II. By the cardinal product $\Pi G_i$ of $G_i$'s we mean the set of all elements $(x_1, \ldots, x_n)$, $x_i$ in $G_i$, where $(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n)$ if and only if $x_i \leq y_i$ for all $i$. The class $\Pi G_i$ becomes a group if we define $(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = (x_1y_1, \ldots, x_ny_n)$ for $x_i$, $y_i \in G_i$. Moreover, $\Pi G_i$ becomes a l. o. c. g. II. In § 2, we deal some properties of l. o. c. g. II. And in § 3, we shall investigate the necessary and sufficient conditions that a l. o. c. g. II is group-isomorphic to a cardinal product of some s. o. c. g. II's.
§ 2 Some properties of l.o.c.g.II's

Let $G$ be a l.o.c.g.II. Throughout this paper, $A$, $B$ and $D$ will denote the set of all conservers, inverters and destroyers, respectively. Evidently, $A^2 \subseteq A$, $AB \subseteq B$, $B^2 \subseteq A$, $A^{-1} \subseteq A$, $B^{-1} \subseteq B$, $AD \subseteq D$, $BD \subseteq D$, hence $D^{-1} \subseteq D$, where $AB$ denotes the set of all elements $ab, (a \in A, b \in B)$. Clearly, by definition, $A, B$ and $D$ are disjoint each other.

The following Lemmas are obvious.

[LEMMA 1] $A$, $A\lor B$ are both subgroups of $G$.

[LEMMA 2] If $x \in A$ and $y \in B$, then

(i) $x(\alpha \lor \beta) = xa \lor x\beta$ for any $\alpha, \beta \in G$ and dually,

(ii) $y(\alpha \lor \beta) = ya \lor y\beta$ for any $\alpha, \beta \in G$ and dually.

[LEMMA 3] Let $G$ be a l.o.c.g.II. $A$ is a dual ideal (or ideal) of $G$ if and only if $e \prec x$ (or $e \succ x$) implies $xeA$. Where $e$ is an identity of $G$.

[PROOF] Assume $A$ is a dual ideal. Then clearly, we see that $e \prec x$ implies $xeA$. Conversely, assume $e \prec x$ implies $xeA$. For $xeA$ and $g \in G$, since $e \prec x \leq x \lor g$, we have $x \lor geA$. And we have $a \lor beA$ for any $a, b \in A$. For, since $a \leq a \lor b$, we have $a \lor beA$. Hence $(a \lor b)^{-1} \leq a^{-1}, b^{-1}$. If $c \leq a^{-1}, b^{-1}$, then we see $(a \lor b)^{-1} \lor c \leq a^{-1}, b^{-1}$. Since $(a \lor b)^{-1} \lor c \leq A, a \lor b \leq [(a \lor b)^{-1} \lor c]^{-1}$, i.e. $(a \lor b)^{-1} \lor c \leq (a \lor b)^{-1}$. Hence $e \leq (a \lor b)^{-1}$, i.e. $(a \lor b)^{-1} = a^{-1} \lor b^{-1}$. Thus we have $a \lor b \in A$, i.e. $A$ is a dual ideal, as desired.

A $< B$ means $a < b$ for all $a \in A$, $b \in B$.

[THEOREM 1] Let $G$ be a l.o.c.g.II. Then

(i) $A, B$ are anti-order isomorphic

(ii) $D$ is the sum of some $dA$'s, where $d$ is a destroyer.

(iii) if $e \prec x$ implies $xeA$, then $A$ is an l-subgroup of $G$.

[PROOF] Let $b$ be an element of $G$. Then $bA = B$. For, since $bA \subseteq BA \subseteq B$, we have $bA \subseteq B$. And $b = bb^{-1}b$ for any $b \in B$. Thus $b \in bA$. Hence $B \subseteq bA$. The mapping $f(a) = ba$ ($a \in A, b$ is a fixed element of $B$) is a one-to-one and anti-order isomorphism by (ii) of Lemma 2, i.e. (i) holds. Since $A$ is a subgroup of $G, G$ is the sum $A \lor bA \lor dA \lor \cdots \lor DA$ (disjoints), where $b \in B$, $d \in D$. Therefore $D$ is the sum of some $dA$'s. (iii) is obvious, by Lemma 1, 2.

A destroyer $d$ is called proper if for any $a < b$ in $G, da \neq db$.

[THEOREM 2] Let $G$ be a po.c.g.II in which any destroyer is proper. Then $A$ and $B$ are convex subsets of $G$. ($a$ subset $S$ of $G$ is convex if $a, \beta \in S$ and $a \prec x \prec \beta$ implies $x \in S$).
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[PROOF] Suppose $\alpha < x < \beta$ ($\alpha, \beta$ in $A$).

(i) If $x \in B$, $ax < \alpha \beta$. Since $\alpha < \beta$ implies $ax > \beta x$, we have $\beta x < \alpha \beta$ i.e. $x < \alpha$.

It is contrary to $\alpha < x$.

(ii) if $x \in D$, $ax < \alpha \beta$. Since $e < \alpha^{-1}x < \alpha^{-1} \beta$ and $\alpha^{-1}x \in D$, we have $\alpha^{-1}x \not= \alpha^{-1} \beta$. while $\alpha^{-1}x < \alpha^{-1} \beta < \alpha^{-1} x \alpha^{-1} \beta$, we have a contradiction. Hence $\alpha x \not\in A$, i.e. $A$ is a convex subset of $G$.

Suppose $a < y < \beta$ ($\alpha, \beta$ in $B$). Since $\alpha^{-1}y \in B$, we have $e > \alpha^{-1}y > \beta \alpha^{-1}$. And $e, \beta \alpha^{-1} \epsilon A$, hence $ya^{-1} \epsilon A$ i.e. $ycB$.

We proceed now to investigate the distributivity of the subset $A \vee B$.

[THEOREM 3] Let $G$ be a l.o.c.g. II in which $A < B$, and $e < x$ implies $xeA$. Then the subset $A \vee B$ is a distributive sublattice of $G$.

[PROOF] Since, by Lemma 3, $A$ is an I-subgroup, and $A$ and $B$ are anti-order isomorphic, we see: $A$ and $B$ are both distributive sublattices of $G$. If $x, y, \epsilon \alpha A \vee B$, and $x \alpha = y \alpha \epsilon A, x \alpha \epsilon = y \alpha \epsilon A$, then we can easily see $x = y$ for any case. Hence $A \vee B$ is a distributive sublattice of $G$.

§ 3 Decomposition of po.c.g. II into s.o.c.g. II's Throughout this section, we assume $B < A$. By a simply ordered commutative group (=s.o.c.g.) we mean a group $G$ satisfying (a) $G$ is simply ordered, and (b) $a \leq b$ ($a, b$ in $G$) implies $ac \leq !c$ for any $c \in G$. [3]. By a simply ordered commutative group of the second kind (=s.o.c.g. II), we mean a group $G$ satisfying (a) $G$ is simple ordered, and (b) every element of $G$ is either a conserver or an inverter.

We can easily see that the set $A$ of all conservers of a s.o.c.g. II becomes a s.o.c.g. Let $B$ denote the set of all inverters of a s.o.c.g. II. And assume also $B < A$ in this section.

Now we shall investigate the condition that po.c.g. II is to be a cardinal product of some s.o.c.g. II's.

Before beginning our study, we state the following Lemma similarly to the way used by A.H. Clifford in [1].

we shall call an element $\epsilon$ unit element in $G$ if $\epsilon^2 = e$.

[LEMMA 4] For given a s.o.c.g. $G$, we can construct a s.o.c.g. II with an inverter unit element.

[PROOF] Let $\rho$ be a convex congruence relation (see [1]) in $G$, and $k$ an element of $G$, such that $x \rho y$ implies $kx = ky$, Since $G_1$ is a group, we see $x = y$ if $x \rho y$. Let $G_2$ be the set of congruence classes of $G_1$ mod $\rho$, and let $\phi$ be the
canonical mapping (see [1]) of \( G_1 \) onto \( G_2 \). The order relation in \( G_2 \) is defined as the followings: \( \phi(x) < \phi(y) \) if and only if \( x > y \) in \( G_1 \).

Let \( G = G_1 \sqcup G_2 \) (disjoint) and order \( G \) so that \( G_2 < G_1 \) Define product in \( G \) as the followings: for \( x, y \in G_1 \), \( x \phi(y) = \phi(xy) \), \( \phi(x) \phi(y) = xy \).

We now show that the above-defined \( G \) is a s.o.c.g. II as desired. To see that \( G \) is a group. We first investigate the associativity of \( G \) with respect to above product; for example

\[
\phi(x)y \cdot \phi(z) = \phi(xy) \cdot \phi(z) = x \cdot yz = \phi(x) \cdot \phi(yz) = \phi(x) \cdot y \phi(x) \\
\phi(x) \phi(y) \cdot \phi(z) = \phi(xy) \phi(z) = \phi(xy \cdot z) = \phi(x) \cdot yz = \phi(x) \cdot \phi(y) \phi(z)
\]

And for any \( \phi(x) \in G_2 \), we have \( \phi(x) \phi(x^{-1}) = e \) i.e. \( \phi(x^{-1}) = (\phi(x))^{-1} \). Hence we see that \( G \) becomes a group with respect to the product of \( G_1 \) and above-defined products. And we easily see that every element of \( G_1 \) is a conserver and every element of \( G_2 \) an inverter in \( G \). And clearly, \( \phi(e) \) is an inverter unit of \( G \).

In the n-dim. Euclidean space, the subset

\[
F_n = \{ (1,1,\ldots,1), (-1,1,\ldots,1), (1,-1,\ldots,1), \ldots, (-1,-1,\ldots,-1) \}
\]

becomes a l.o.c.g. II of order \( 2^n \), if we define product and order relations in \( F_n \) as followings:

\[
(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) = (a_1b_1, \ldots, a_nb_n) \\
(a_1, \ldots, a_n) \leq (b_1, \ldots, b_n) \text{ if and only if } a_i \leq b_i \text{ for all } i.
\]

Clearly, the element \((-1,-1,\ldots,-1)\) is an inverter unit element of \( F_n \). We call \( F_n \) fundamental l.o.c.g. II.

[THEOREM 4] Let \( G \) be a p.o.c.g. II with an inverter unit element. And let \( G \) be group-isomorphic to a cardinal product of \( n \) s.o.c.g. II's with an inverte unit element. Then

(i) The set \( A \) of all conservers of \( G \) is group-isomorphic to a cardinal product of \( n \) s.o.c.g.'s, and \((G:A) = 2^n\).

(ii) There exists a subgroup \( \mathcal{Y}_n = \{ f_1(e), f_2, \ldots, f_3^n \} \) of \( G \) such that \( \mathcal{Y}_n \) is isomorphic to \( F_n \), and \( f_i f_j \notin A \) (\( i \neq j \)).

Conversely, if (i) and (ii) hold in \( G \), then we can construct a l.o.c.g. II which is a cardinal product of \( n \) s.o.c.g. II's such that is group-isomorphic to \( G \).
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[PROOF] Let $G$ be a cardinal product ($=\Pi G_i$) of $n$ s.o.c.g. II $G_i$'s. Let $A_i$ be the set of all conservers of $G_i$. Then we easily see that $A$ is cardinal product $\Pi A_i$ of $A_i$'s. Since $A_i$ is a s.o.c.g., $A$ is the cardinal product of $n$ s.o.c.g.'s. Let $\varepsilon_i, e_i$ be an inverter unit, an identity of $G_i$, respectively. Then we easily see that $A$ is cardinal product $\Pi A_i$ of $A_i$'s. Since $A_i$ is s.o.c.g., $A$ is the cardinal product of $n$ s.o.c.g.'s. Let $\varepsilon_i, e_i$ be an inverter unit, an identity of $G_i$, respectively. Then we easily see that $A$ is cardinal product $\Pi A_i$ of $A_i$'s. Since $A_i$ is a s.o.c.g., $A$ is the cardinal product of $n$ s.o.c.g.'s. Let $\varepsilon_i, e_i$ be an inverter unit, an identity of $G_i$, respectively. Then we easily see that $A$ is cardinal product $\Pi A_i$ of $A_i$'s. Since $A_i$ is s.o.c.g., $A$ is the cardinal product of $n$ s.o.c.g.'s.

Let $\mathcal{Y}_n = \{(a_1, \ldots, a_n) | a_i = e_i \text{ or } \varepsilon_i\}$ of $\Pi G_i$ is a sub-l.o.c.g. II of order $2^n$, and moreover $\mathcal{Y}_n$ is isomorphic to $F_n$. If $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are two distinct elements of $\mathcal{Y}_n$, then $(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) \in \Pi A_i$. Since $(a_1, \ldots, a_n)^{-1} = (a_1, \ldots, a_n)$ for any $(a_1, \ldots, a_n) \in \mathcal{Y}_n$, $(a_1, \ldots, a_n) \equiv (b_1, \ldots, b_n)$ for mod $\Pi A_i$. On the other hand, if $(x_1, \ldots, x_n) \in \Pi G_i$, then there exists some elements $(a_1, \ldots, a_n)$ of $\mathcal{Y}_n$ such that $(x_1, \ldots, x_n) = (a_1, \ldots, a_n)$ for mod $A$. Therefore $(G: A) = 2^n$. Hence (i) and (ii) hold in $G$.

Conversely, Assume (i) and (ii) hold in given p.o.c.g. II $G$. By (ii), the set $A$ of $G$ is group-isomorphic to a cardinal product $\Pi A_i$ of $n$ s.o.c.g. $A_i$'s. By Lemma 5, we can construct a s.o.c.g. II $G_i$ with an inverter unit element from each $A_i$. Now we must prove that $G$ is group-isomorphic to the cardinal product $\Pi G_i$ of $G_i$'s. To see this, by the foregoing way, we make $\mathcal{Y}_n$ of $\Pi G_i$, so that $\mathcal{Y}_n \cong F_n$. By (ii), $\mathcal{Y}_n$ of $G$ is isomorphic to $F_n$. Thus $\mathcal{Y}_n \cong \mathcal{Y}_n$. Since $G = A \vee A \vee \cdots \vee f_n A$, where $f_1 \in \mathcal{Y}_i$, and $\Pi G_i = (\Pi A_i) \vee f_1 (\Pi A_i) \vee \cdots \vee f_n (\Pi A_i)$, where $f_i \in \mathcal{Y}_i$, the mapping: $f_i a \rightarrow f_i a$ is a group-isomorphism of $G$ onto $\Pi G_i$ ($a$ in $A_i$ in $\Pi A_i$). Where $f_i$ corresponds to $f_i$ by $\mathcal{Y}_n \cong \mathcal{Y}_n$, and $a$ corresponds to $a$ by $A \cong \Pi A_i$. For, if $f_i b \rightarrow f_i b$ ($b$ in $A_i$ in $\Pi A_i$), then $(f_i a) \cdot (f_i b) = f_i a b = f_i c \rightarrow f_i c = f_i f_i a b = (f_i a) \cdot (f_i b)$, where $f_i$ and $f_j$ are in $\mathcal{Y}_n$, $a b = c$ in $A_i$. Clearly, the mapping is one-to-one. Hence $G$ is group-isomorphic to $\Pi G_i$.

§ 4 Examples

[EXAMPLE 1] Let $E_n = \{(a_1, \ldots, a_n) | a_i(\neq 0) \text{ is a real number}\}$. And we define order and product relations in $E_n$ as the followings:
Then $E_n$ is a l.o.c.g. II with destroyers which are not proper.

[EXAMPLE 2] In the $E_n$, if we define order relation as followings $(a_1, \ldots, a_n) \leq (b_1, \ldots, b_n)$ if and only if either $a_i = b_i$ for all $i$ or $a_i < b_i$ (but $a_i \neq b_i$) for all $i$. Then $E_n$ is a l.o.c.g. II. And every destroyer is proper.

[EXAMPLE 3] Let $G$ be the set of all one valued real functions $f(x)$ and it's inverse function $f^{-1}(x)$ defined on $[0,1]$ which are having at most finite number discontinuous points, and $f(x) \equiv 0$ and $f^{-1}(x) \equiv 0$ for all $x \in [0,1]$. Then $G$ becomes a group under ordinary product of functions. Moreover one defines the order in $G$ such that $f(x) \leq g(x)$ means $f(x) \leq g(x)$ for all $x$ on $[0,1]$. Then $G$ is a l.o.c.g. II with destroyers which are not proper.

[EXAMPLE 4] In the $G$ of example 3, one defines the order in $G$ such that $f(x) \leq g(x)$ means either $f(x) = g(x)$ or $f(x) < g(x)$ for all $x$ in $[0,1]$. Then every destroyer in $G$ is proper.

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