ON A CONTINUOUS MAPPING BETWEEN PARTIALLY ORDERED SETS WITH SOME TOPOLOGY

By Tae Ho Choe

1. Introduction and Notations

Let $P$ be a partially ordered set. By the interval topology of $P$, we mean that defined by taking the closed intervals $[a, b]$, $[-\infty, a]$, and $[a, \infty]$ of $P$ as a sub-base of closed sets. Let $f$ be a mapping of a partially ordered set $P$, into another partially ordered set $P_n$. In this paper, we first obtain a necessary and sufficient condition that $f$ be a continuous in their interval topologies. This condition, stated in theorem 1, can be applied to show that if $f$ is a complete isotone of a complete lattice into a complete lattice, then $f$ is a continuous in their interval topologies.

N. Funayama [2] has introduced an imbedding operator $\phi$ in the family of subsets of $P$, and has defined a completion $P_\phi$ of $P$ by the imbedding operator $\phi$. And he has obtained a lot of interesting results that $P$ is imbedded into some complete lattice. In theorem 2 we consider conditions under which $P$ is continuously imbedded into a complete lattice with respect to their interval topologies.

T. Naito [3] has introduced the concept of CP-ideal topology. In §3, we shall deal with similar results of §2 with respect to CP-ideal topology.

We shall use $I$, $I_\alpha$, $I_{\alpha \beta}$, $J$, $J_\alpha$, $J_{\alpha \beta}$ to denote closed intervals in §2 and to denote CP-ideals or dual CP-ideals in §3. We denote the join and the meet of two elements $x$ and $y$ of a lattice by $x \vee y$ and $x \wedge y$ respectively, the join and the meet of all elements of a set $\{a_\alpha | \alpha \in \Delta\}$ by $\sup_{\alpha \in \Delta} a_\alpha$ and $\inf_{\alpha \in \Delta} a_\alpha$ respectively. $A \cup B$ and $\vee_{\alpha \in \Delta} X_\alpha$ will be used to denote the set union of two sets $A$ and $B$, and of sets of the family $\{X_\alpha | \alpha \in \Delta\}$, and $A \cap B$ and $\wedge_{\alpha \in \Delta} X_\alpha$ are the set intersections of them.

Finally, the complement of a set $A$ will be denoted by $A^c$.

2. Interval topology.

We here note that if a subset $S$ of $P$ is a closed set in it's interval topology, then $S$ may be expressed as an intersection of the sets which are unions of a finite number of closed intervals in $P$:
\[ S = \bigwedge_{a, \beta} \{ \bigvee_{\alpha} I_{\alpha\beta} \} \]

where \( I_{\alpha\beta} \) is the form of \([a, b], [a, +\infty], \) or \([-\infty, b] \). Thus an open subset \( O \) in \( P \) is expressed as

\[ O = \bigvee_{a, \beta} \{ \bigwedge_{\alpha} I_{\alpha\beta}^c \} \]

Let \( P \) be a partially ordered set. A subset \( S \) of \( P \) is called to be covered by a finite closed intervals of \( P \) if there exist a finite number of closed intervals \( I_n \) such that \( S \subseteq \bigcup_n I_n \).

We first prove the following theorem:

**THEOREM 1.** Let \( P_1 \) and \( P_2 \) be two partially ordered sets, and \( f \) a mapping of \( P_1 \) into \( P_2 \). \( f \) is continuous in their interval topologies if and only if for any closed interval \( J \) of \( P_2 \) and any element \( x \) of \( P_1 \) such that \( x \in f^{-1}(J) \), there exists a covering of \( f^{-1}(J) \) by means of a finite number of closed intervals none of which contains \( x \).

**PROOF.** Suppose that \( f \) is a continuous mapping of \( P_1 \) into \( P_2 \). And \( x \in f^{-1}(J) \) for a closed interval \( J \) of \( P_2 \) and an element \( x \) of \( P_1 \). Since \( f^{-1}(J) \) is a closed set in \( P_1 \), it may be expressed as following \( f^{-1}(J) = \bigwedge_{\alpha} \{ \bigvee_{\beta} I_{\alpha\beta} \} \), where \( I_{\alpha\beta} \) is a closed interval in \( P_1 \). Thus \( x \in \bigvee_{\beta=1}^{n_{\alpha}} I_{\alpha\beta} \) for some \( \alpha \). Moreover \( f^{-1}(J) \subseteq \bigvee_{\alpha} I_{\alpha\beta}^c \) and \( x \in I_{\alpha\beta}^c \) (\( 1 \leq \beta \leq n_{\alpha} \)). Conversely, for an element \( x \) of \( P_1 \), let \( O_2 \) be a neighborhood of \( f(x) \) in \( P_2 \). It suffices to show that for some open subset \( O \) containing \( x \), \( O \subseteq f^{-1}(O_2) \). Thus we may assume that \( O_2 \) is an open set in \( P_2 \), which may be expressed as \( O_2 = \bigvee_{\beta=1}^{n_{\alpha}} I_{\alpha\beta}^c \), where \( J_{\alpha\beta} \) is a closed interval or the empty set or \( P_2 \). And there exists a closed intervals \( J_{\alpha\beta} \) such that \( f(x) \in J_{\alpha\beta} \), i.e., \( x \in f^{-1}(J_{\alpha\beta}) \) for some \( \alpha \) and all \( \beta \) corresponding to \( \alpha \). By the hypotheses, there are a finite number of closed intervals \( I_{\alpha}^{\beta} \) (\( \alpha \)) such that \( f^{-1}(J_{\alpha\beta}) \subseteq \bigvee_{\beta} I_{\alpha\beta}^{\beta} \), i.e., \( \bigvee_{\alpha} I_{\alpha\beta}^{\beta} \subseteq f^{-1}(J_{\alpha\beta}) \) for each \( \beta \). On the other hand, \( x \in \bigwedge_{\beta=1}^{n_{\alpha}} (\bigvee_{\alpha} I_{\alpha\beta}^c) \subseteq \bigwedge_{\beta=1}^{n_{\alpha}} f^{-1}(J_{\alpha\beta}^c) \subseteq f^{-1}(O_2) \), which completes the proof.
A continuous mapping

A mapping $f$ of a partially ordered set $P_1$ into $P_2$ is called a complete isotone if $\sup_{x \in d} x \leq \sup_{x \in d} f(x)$ exist and $x = \sup_{x \in d} x$ implies $f(x) = \sup_{x \in d} f(x)$, and it's dual. Theorem 1 can be applied to show the following

**COROLLARY 1.** Let $f$ be a complete isotone of a complete lattice $P_1$ into a complete lattice $P_2$. Then $f$ is continuous in their interval topologies.

**PROOF.** Let $J$ be a closed interval in $P_2$ and $x$ an element in $P_1$ such that $x \notin f^{-1}(J)$. We shall show that there is a closed interval $I$ in $P_1$ not containing $x$ such that $f^{-1}(J) \subseteq I$. If the set $S = \{y \in P_1 | f(y) \in J\}$ is empty, then we may take the empty set as $I$. Therefore we may assume that $S$ is non-empty. Let $a = \inf S$, $b = \sup S$. If we suppose $x \in [a, b]$, then $f(x) \in J$ because $f$ is a complete isotone. It follows that $x \in f^{-1}(J)$ which is contrary. Clearly we see that $f^{-1}(J) \subseteq [a, b]$, which completes the proof.

N. Funayama [2] has defined an imbedding operator $\phi$ in the family of subsets of a partially ordered set $P$. $A$ is called $\phi$-closed if $\phi(A) = A$. All the $\phi$-closed sets form a complete lattice $P_\phi$ under set inclusion, $P_\phi$ is called the completion of $P$ by the imbedding operator $\phi$. And he has proved that if a collection $\Omega = \{A_\lambda\}$ of subsets of $P$ satisfies the following conditions: (i) every $A_\lambda$ is an ideal of $P$, i.e., $\lambda \in A_\lambda$ and $x \leq a$ then $x \in A_\lambda$, (ii) every principal ideal is a member of $\Omega$, (iii) $\Omega$ is M-complete, i.e., for any subset $\{B_\lambda\}$ of $\Omega$, $\wedge B_\lambda \in \Omega$, (iv) $P \in \Omega$, then there exists an uniquely determined imbedding operator $\phi$ on $P$ such that $\Omega = P_\phi$.

The theorem 2 of [2] says that let $\phi$ be an imbedding operator on $P$, then $P$ is imbedded into $P_\phi$ by $f : f(a) = \langle a \rangle$ (=principal ideal generated by $a$), where $f$ is $\phi$-isomorphism, i.e., $f(a) \geq f(b)$ if and only if $a \geq b$.

The lemma 2 and theorem 2 of [2] and theorem 1 give us the following lemma:

**LEMMA 1.** Let $P$ be a partially ordered set. If there is a collection $\Omega$ satisfying (i)~(iv) in $P$, and if the mapping $f : f(a) = \langle a \rangle$ of $P$ into $\Omega$ satisfies the hypothesis of theorem 1, then $P$ is continuously imbedded into the complete lattice $P_\phi$ ($= \Omega$).

Hence, by above lemma 1 and corollarily, we have

**THEOREM 2.** Under the hypotheses of lemma 1, if $g$ is a complete isotone
of \( P \) into a complete lattice \( L \), then \( P \) is continuously imbedded in \( L \) by \( g \cdot f \) in to \( L \) in their interval topologies.

3. CP-ideal topology

In this section, we denote \( P \) to be a lattice. An ideal \( I \) is said to be a prime ideal if and only if \( x \cap y \) implies \( x \in I \) or \( y \in I \). A prime ideal \( I \) is called a CP-ideal if and only if the following condition holds: if \( \{ x_\alpha | \alpha \in A \} \subseteq I \) and there exists \( \sup x_\alpha \), then \( \sup x_\alpha \in I \). Dually, a dual prime ideal and a dual CP-ideal are defined (T. Naito [3]). The union of \{ all CP-ideals of \( P \) \}, \{ all dual CP-ideals of \( P \) \} and \( \emptyset \) is denoted by \( \mathcal{L}P \), where \( \emptyset \) is the empty set. We recall that the CP-ideal topology of a lattice \( P \) is that defined by taking the elements of \( \mathcal{L}P \) as a sub-base of closed sets of the space \( P \).

In the same way as in §2, We can prove the following

THEOREM 3. Let \( P_1 \) and \( P_2 \) be two lattices, and \( f \) is mapping of \( P_1 \) into \( P_2 \). \( f \) is continuous in their CP-ideal topologies if and only if for any member \( J \) of \( \mathcal{L}P \) of \( P_2 \) and any element \( x \) of \( P_1 \) such that \( x \in f^{-1}(J) \) there exists a covering of \( f^{-1}(J) \) by means of a finite number of members of \( \mathcal{L}P \) none of which contains \( x \).

As a corollary of the theorem 3, we also have

COROLLARY 2. Let \( f \) be a complete isotone of a complete lattice \( P \), into a complete lattice \( P_2 \). Then \( f \) is a continuous mapping of \( P_1 \) into \( P_2 \) in their CP-ideal topologies.

PROOF. Let \( J \) be a member of \( \mathcal{L}P \) of \( P_2 \) and \( x \) an element in \( P_1 \) such that \( x \in f^{-1}(J) \). We shall show that there exists a member \( I \) of \( \mathcal{L}P \) of \( P_1 \) not containing \( x \) such that \( f^{-1}(J) \subseteq I \). We consider \( J \) into three cases:

(i) \( J \) is a nonvoid CP-ideal. Let \( S = \{ y_r | f(y_r) \in J \} \). If \( S = \emptyset \), i.e. \( f^{-1}(J) = \emptyset \) we then take the empty set as \( I \). And we may assume \( S \neq \emptyset \). Set \( a = \sup S \). Then \( \langle a \rangle \) is a CP-ideal of \( P_1 \). For, if \( u \cap v \in \langle a \rangle \), then \( f(u) \cap f(v) \subseteq f(u) \cap f(v) = J \). Thus we have either \( f(u) \in J \) or \( f(v) \in J \), i.e. \( u \in \langle a \rangle \) or \( v \in \langle a \rangle \). It follows that \( \langle a \rangle \) is a prime ideal. And if \( \{ x_\alpha | \alpha \in A \} \) and there exists \( \sup x_\alpha \), then clearly \( \sup x_\alpha \in \langle a \rangle \). Moreover we can see easily: \( x \in \langle a \rangle \) and \( f^{-1}(J) \subseteq \langle a \rangle \).

(ii) \( J \) is a nonvoid dual CP-ideal. This is a dual of (i).
A continuous mapping

(iii) \( J = \phi \). In this case, we may take the empty set as \( I \). This proves our corollary.

We recall (Funayama, [2]) that if a partially ordered set \( P \) is imbedded in a complete lattice \( L \) by a mapping \( \theta \), \( \theta \) is called \( J \)-density if any element \( x \) in \( L \) can be represented as a join of elements of \( \theta(P) \), that is \( x = \sup \theta(a_r) \), where \( a_r \in P \). And in [2], he noted that if \( P \) is imbedded in \( L \) \( J \)-densely by \( \theta \), then \( \theta(a) = \inf_{r} \theta(a_r) \) in \( L \) if and only if \( a = \inf_{r} a_r \) in \( P \).

**Lemma 2.** Let a lattice \( P \) be imbedded in a complete lattice \( L \) \( J \)-densely by \( \theta \). Suppose that \( \{x_a | a \in A\} \subset P \) and there exists \( a = \sup x_a \) then \( \theta(a) = \sup_{a \in A} \theta(x_a) \).

Then \( \theta \) is a continuous mapping of \( P \) into \( L \) in their CP-ideal topologies.

**Proof.** It is sufficient to show that for some CP-ideal \( J \) of \( L \), \( S = \{x \in P | \theta(x) \in J\} \) is also a CP-ideal of \( P \). In fact, clearly \( S \) is a prime ideal of \( P \). And if \( \{x_a | a \in A\} \subset S \) and there exists \( \sup x_a \) in \( P \), then we have \( \sup x_a \in S \) because \( \theta(\sup x_a) = \sup_{a} \theta(x_a) \in J \). Hence \( S \) is a CP-ideal of \( P \). And dually.

Theorem 2 of [2] and lemma 2 give us the following

**Theorem 4.** Let \( \phi \) be an imbedding operator on a lattice, and \( \phi^*: \phi^*(a) = (a) \) be the mapping of \( P \) into \( P \) such that \( \phi^*(x) = \sup_{a \in A} \phi^*(x_a) \) if \( x = \sup_{a \in A} x_a \) exists.

And if \( f \) be a complete isotope of \( P \) into a complete lattice \( L \), then \( P \) is continuously imbedded into \( L \) in their CP-ideal topologies.

June, 1962
Mathematical Department
Kyungpook University
Taegu, Korea

**References**

