DIRECT DECOMPOSITIONS OF LATTICES OF CONTINUOUS FUNCTIONS

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Introduction. If $X$ is a topological space and if $K$ is a chain equipped with its order topology, then we denote by $C(X, K)$ the lattice of all continuous functions from $X$ to $K$. In the paper [I] R.L. Blair and C.W. Burill proved following theorem.

THEOREM B. Let $X$ be a topological space, let $K$ be a chain equipped with its order topology, and let $L$ be an adequate sublattices of $C(X, K)$. If $L$ is isomorphic to the direct product of two lattices $L_1$ and $L_2$, then $X$ is the union of disjoint open and closed sets $X_1$ and $X_2$ having the property that $L_i$ is isomorphic to $L_{X_i}$ ($i=1, 2$). Moreover, $X_i$ is non empty if and only if $L_i$ has only two distinct elements.

And in the paper they defined an adequate sublattice as following.

DEFINITION. A sublattice $L$ of $C(X, K)$ will be called adequate in case for each $x \in X$ there exist functions $f, g \in L$ such that $f(x) > g(x)$.

In the paper Page 634 they stated following problem.

PROBLEM. What are necessary and sufficient conditions on a sublattice $L$ of $C(X, K)$ in order that a direct decomposition of $L$ be reflected in a corresponding decomposition of $X$? In any case, the hypothesis of adequacy cannot simply be deleted.

So in this paper I modify condition adequate as sub-adequate (defined in this paper) and prove THEOREM 1, which is a little expansion of THEOREM B. Also, if $X$ is a topological space such that for each $x \in X$ there exists the least open and closed set containing it, we prove THEOREM 2, in which we use hypothesis of r-adequacy which is weaker than sub-adequacy. The technic used for the proof of THEOREM 1, is similar with the technic which is used in the proof of THEOREM B, hence it can be obtained easily if we use THEOREM B,
but for the convenience of reader I prove it independently.

Theorems.

For the proof of theorems, we include here some properties of lattice which can be found in [1] or [2]. By a prime ideal of a lattice $L$ we mean a non empty proper subset $P$ of $L$ such that (i) if $a, b \in P$ then $a \cup b \in P$ and (ii) $a \cap b \in P$ if and only if $a \in P$ or $b \in P$. A dual prime ideal is the complement of a prime ideal. If $L_1$ and $L_2$ are lattices and $P$ is a prime ideal of the direct product $L_1 \times L_2$, then either $P = P_1 \times L_2$ for some prime ideal $P_1$ of $L_1$ or $P = L_1 \times P_2$ for some prime ideal $P_2$ of $L_2$. If $Y$ is a subset of $X$ and if $f \in C(X, K)$, then $f \mid Y$ denotes the restriction of $f$ to $Y$. If $L$ is a sublattice of $C(X, K)$ then we set $L_Y = \{ f \mid Y : f \in L \}$. It is clear that $L_Y$ is a sublattice of $C(Y, K)$. We define the terminology which is used for the proof of theorems.

**DEFINITION.** A non void subset $M$ of $X$ will be called as a cutting set of $X$ if and only if there exist two subsets $X_1, X_2$ of $X$ such that $X_1 \cap X_2 = \phi$ (void), $X_1 \cup X_2 = X$, and $X_1 \cap X_2 = M$. ($\overline{X}_i$ denotes the closure of $X_i$).

**DEFINITION.** If $L$ is a sublattice of $C(X, K)$ with following two conditions, then we call $L$ a sub-adequate sublattice.

1. For a dense subset $D$ of $X$ and for each $x \in D$ there exist functions $f, g \in L$ such that $f(x) > g(x)$.
2. Let $M$ be a cutting set of $X$ such that $X_1 \cap X_2 = \phi$, $\overline{X}_1 \cup \overline{X}_2 = X$ and $\overline{X}_1 \cap \overline{X}_2 = M$. If $f(x) = g(x)$ for each pair $(f, g) \in L$, for each $x \in M$ then there exist two functions $f_1 \in L_{\overline{X}_1}$ and $g_2 \in L_{\overline{X}_2}$ such that the extension of $(f_1, g_2)$ on $X$ is not contained in $L$. (Extension of $(f_1, g_2)$ means the function $h(x) = f_1(x)$ if $x \in \overline{X}_1$ and $h(x) = g_2(x)$ if $x \in \overline{X}_2$).

**THEOREM 1.** Let $X$ be a topological space, let $K$ be a chain equipped with its order topology, and let $L$ be a sub-adequate sublattice of $C(X, K)$. If $L$ is isomorphic to the direct product of two lattices $L_1$ and $L_2$, then $X$ is the union of open and closed subsets $X_1$ and $X_2$ having the property that $L_i$ is isomorphic to $L_{\overline{X}_i}$ ($i = 1, 2$). Moreover, $X_i$ is non empty if and only if $L_i$ has two distinct elements.

**PROOF.** Since $L$ is sub-adequate, there is a dense subset $D$ such that $x \in D$
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If and only if \( f(x) \geq g(x) \), for some pair \((f, g) \in L\). If \( x \in D\) and \( f \in L\), we set
\[
P_x(f) = \{ g \in L : g(x) \leq f(x) \}
\]
and
\[
P^x(f) = \{ g \in L : g(x) \geq f(x) \}
\]
It is clear that \( P_x(f) \) (resp \( P^x(f) \)) is a prime ideal of \( L \) provided that it is a proper subset of \( L \). Since \( x \in D \), we have either \( P_x(f) \) is a prime ideal of \( L \) or \( P^x(f) \) is a dual prime ideal of \( L \). We choose an isomorphism \( \delta \) from \( L \to L \times L \) and fix an element \( k \in L \). Denote by \( \mathcal{P}_i \) (resp \( \mathcal{P}_2 \)) the collection of all prime ideals of \( L \). For \( i = 1, 2 \), denote by \( D_i \) the set of all \( x \in D \) such that either \( P_x(k) \in \mathcal{P}_i \) or \( L-P^x(k) \in \mathcal{P}_i \). Then it is easily seen that \( D_1 \) and \( D_2 \) are disjoint and that \( D = D_1 \cup D_2 \). We shall show that \( D_1 \cap D_2 = D_1 \cap D_2 = \emptyset \). For \( D_1 \cap D_2 = \emptyset \), if \( y \in D_1 \) and \( y \notin D_2 \) then clearly \( y \notin D_2 \). In the case \( y \in D_1 \) and \( y \notin D \), then
\[
\cap \{ P_x(k) : x \in D_1 \} \subseteq P_y(k)
\]
and
\[
\cap \{ P^x(x) : x \in D_1 \} \subseteq P^y(k)
\]
from this we have \( y \notin D_2 \).

Hence we have \( D_1 \cap D_2 = \emptyset \) and similarly \( D_1 \cap D_2 = \emptyset \). We know that
1. \( D \) is a dense subset of \( X \),
2. \( D = D_1 \cup D_2 \), \( D_1 \cap D_2 = \emptyset \), \( D_1 \cap D_2 = \emptyset \) and
3. \( D_1 \cap D_2 = D_1 \cap D_2 = \emptyset \) and
(4) \( D_1 \cap D_2 = D_1 \cap D_2 = X \).
Let us suppose \( M(M \in D_1 \cap D_2) \) is not a void set, then we have \( M \subseteq X - D \) and \( M \) is a cutting set. Hence by hypothesis of sub-adequacy there exist two functions \( f, g \in L \) such that \( f \in L \), \( g \in L \), \( f \in L \), \( g \in L \) and \( f \notin L \) and the extension of \( f, g \) on \( X \) is not contained in \( L \). Now \( \pi_i \) be the projection of \( L_1 \times L_2 \) onto \( L_1 \) and consider the mapping \( \phi_i \in \mathcal{P}_i \cdot \delta \) from \( L \) onto \( L_1 \). Let \( f, g \in L \) and suppose that \( \phi_i(f) \leq \phi_i(g) \) but \( f(x) > g(x) \) for some \( x \in D_1 \). Then \( P_x(g) \) is a prime ideal of \( L \) that contain \( g \) but not \( f \). If \( P_x(k) \approx L \), then, since \( P_x(g) \cap P_x(k) \) contains a prime ideal \( P_x(g \cap k) \). \( P_x(g) \) must map onto \( P_1 \times L_2 \) for some prime ideal \( P_1 \) of \( L_1 \). But then \( \phi_i(f) \in P_1 \) so that \( f \in P_x(g) \) a contradiction. Moreover, if \( P_x(k) \approx L \), then a dual argument again yields a contradiction. Arguing similarly for \( D_2 \), we therefore conclude that

1. \( \phi_i(f) \leq \phi_i(g) \) implies \( f \upharpoonright D_i \leq g \upharpoonright D_i \) \((i = 1, 2)\)

and

2. \( \phi_i(f) \leq \phi_i(g) \) implies \( f \upharpoonright D_i \leq g \upharpoonright D_i \) \((i = 1, 2)\).

Now suppose that \( f \upharpoonright D_i \leq g \upharpoonright D_i \) but that \( \phi_i(f) \leq \phi_i(g) \). Since \( L_1 \) is distributive,
Zorn's lemma provides a prime ideal \( P \) in \( L_1 \) that contains \( \phi_i(f) \) but not \( \phi_i(f) \). Let \( P \) be the prime ideal in \( L \) that maps onto \( P_1 \times L_2 \). Then \( g \in P \) and \( f \notin P \). Let \( h=\delta^{-1}(\phi_1(f), \phi_1(g)) \) so that \( h \notin P \). Now \( \phi_2(h)=\phi_2(g) \) and therefore by (1), \( h|D_2=g|D_2 \). But then \( f \cap h \subseteq g \) so that \( f \cap h \in P \), a contradiction. Using a similar argument for \( \phi_2 \) we obtain

\[
(3) \quad f|D_i \subseteq g|D_i \text{ implies } \phi_i(f) \subseteq \phi_i(g) \quad (i=1, 2)
\]

and

\[
(4) \quad f|\overline{D}_i \subseteq g|\overline{D}_i \text{ implies } \phi_i(f) \subseteq \phi_i(g) \quad (i=1, 2).
\]

We conclude from (2) or (3) that \( \phi_i : f|D_i \rightarrow \phi_i(f) \) is a well defined map from \( L|D_i \) onto \( L_1 \). Moreover by (1) or (2) \( \phi_i \) is one to one and \( \phi_i^{-1} \) is also order preserving. Hence \( \phi_i \) is an isomorphism. (***) Now we prove \( M=\overline{D}_1 \cap \overline{D}_2 \) must be a void set. If \( M \) is not a void set, then there exist functions \( f_1, g_2 \) such that \( f_1 \in L|D_1, g_2 \in L|D_2 \) and the extension of \( (f_1, g_2) \) on \( X \) is not contained in \( L \). Where we know \( \phi_i \) is an isomorphism between \( L|D_i \) and \( L_1 \), and \( \phi_2 \) is an isomorphism between \( L|D_2 \) and \( L_2 \). Let \( \phi_i(f_1)=m_1 \) then \( m_1 \in L_1 \), and let \( \phi_2(g_2)=m_2 \) then \( m_2 \in L_2 \). Hence if we set \( h=\delta^{-1}(m_1, m_2) \) then \( h \in L \). It follows that \( h|\overline{D_i}=f_i \) and \( h|\overline{D_2}=g_2 \), that is the extension of \( (f_1, g_2) \) on \( X \) is contained in \( L \). Thus we have a contradiction, hence \( M \) is a void set. We have proved that \( \overline{D}_1 \cap \overline{D}_2=\phi \) and \( X=\overline{D}_1 \cup \overline{D}_2 \). If \( L|D_i \) has two distinct elements then there exist functions \( f, g \in L \) such that \( f|D_i \cong g|D_i \), hence there exists a point \( x_i \in D_i \) such that \( f(x_i) \cong g(x_i) \). Thus we have \( D_i \cong \phi \). Conversely if \( D_i \) is not void then \( D_i \) is not void, hence there is a point \( x \in D_i \) and \( f(x_i)>g(x_i) \) for a pair \( f, g \in L \). Thus we have \( L|D_i \) contains more than two elements. If we set \( X_1=D_1 \) and \( X_2=D_2 \) then we have the decomposition we need. We proved THEOREM 1.

Now we define some terminologies which will be used in the proof of THEOREM 2.

**DEFINITION.** A topological space \( X \) will be called as a \( L.O.C.-space \) if for each \( x \in X \) there exists the least open and closed set containing \( x \), and if \( N \) is the least closed and open set containing a point \( x \), then we call \( N \) a \( l.o.c.-set \).

We can see a discrete space or a topological space which has only finite number of open and closed sets is a \( L.O.C.-space \).

**DEFINITION.** If \( L \) is a sublattice of \( C(X, K) \) with following two conditions, then we call it a \( r\)-adequate sublattice.
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(1) If \( N \) is a l.o.c.-set of \( X \) then \( L \) contains some functions \( f, g \) such that \( f(x) > g(x) \) for some \( x \in N \).

(2) Let \( M \) be a cutting set of \( X \) such that \( X_1 \cap X_2 = \emptyset \), \( X_1 \cup X_2 = X \) and \( X_1 \cap X_2 = M \). If \( f(x) = g(x) \) for each pair \( f, g \in L \) and for each \( x \in M \), and if \( L_{X_1} \) and \( L_{X_2} \) both have more than two elements, then there exist two functions \( f_1 \in L_{X_1} \) and \( g_2 \in L_{X_2} \) such that the extension of \( (f_1, g_2) \) on \( X \) is not contained in \( L \).

THEOREM 2. Let \( X \) be a L.O.C.-space, let \( K \) be a chain equipped with its order topology, and let \( L \) be a r-adequate sublattice of \( C(X, K) \). If \( L \) is isomorphic to the direct product of \( L_1 \) and \( L_2 \) (both have more than two elements), then \( X \) is the union of open and closed subsets \( X_1 \) and \( X_2 \) having the property that \( L_i \) is isomorphic to \( L_{X_i} \). Moreover, \( X_i \) is non-empty.

PROOF. Let \( D = \{ x \in X : f(x) > g(x) \} \) for some \( \{ f, g \in L \} \). Then we know \( D \) is not void set, we consider \( D \) as a topological space equipped with relative topology. Clearly \( L_D \) is a sub-adequate sublattice of \( C(X, K) \). We denote isomorphism by \( \cong \). Then \( L \cong L_D \), suppose \( L \cong L_1 \times L_2 \), then we have \( L_D \cong L_1 \times L_2 \). By THEOREM 1, \( D \) is decomposed into \( D_1, D_2 \) such that \( D = D_1 \cup D_2 \). \( D_i \) are open and closed in the space \( D \) and \( L_{D_i} \cong L_i \).

We set

\[ \alpha = \{ N_\alpha : N_\alpha = \text{l.o.c.-set of } X \}, \]

\[ \alpha_1 = \{ N_\alpha : N_\alpha \cap D_2 = \emptyset, \ N_\alpha \in \alpha \}, \]

and

\[ \alpha_2 = \{ N_\alpha : N_\alpha \cap D_2 = \emptyset, \ N_\alpha \in \alpha \}. \]

Let

\[ N_1 = \{ x : x \in X \setminus D, \ x \in N_\alpha \text{ for some } N_\alpha \in \alpha_1 \}, \]

and

\[ N_2 = \{ x : x \in X \setminus D, \ x \in N_\alpha \text{ for some } N_\alpha \in \alpha_2 \}. \]

Let \( X_1 = D_1 \cup N_1 \), and \( X_2 = N_2 \cup D_2 \).

We shall show \( X_1 \cup X_2 = X \) and \( X_i (i = 1, 2) \) are two disjoint open and closed sets of \( X \). By the construction it is clear that \( X_1 \cup X_2 = X \) and \( X_1 \cap X_2 = \emptyset \). By the definition of \( D \), for each \( x \in D \), there exist two functions \( f, g \in L \) such that \( f(x) \)
Since \( f, g \) are continuous functions, there is an open set \( v_x \) of \( x \), such that for each \( x\alpha \in v_x, f(x\alpha) > g(x\alpha) \), hence we have \( v_x \subset D \). Thus \( D \) is an open set of \( X \), and hence \( D \cap N \) are open sets of \( X \). Now let \( x \in X \), if \( x \in D \) then \( D \) is an open neighborhood of \( x \) in \( X \), and if \( x \in N \), then by the construction of \( N \), there is an \( \text{l.o.c.-} \) set \( N_x \) which contains \( x \), and \( N_x \) is contained in \( X \), and this \( N_x \) is an open set of \( X \). Hence we have \( X \) is an open set of \( X \). Since \( D \) is an open set in \( X \), we have \( D \) (complement of \( D \) in \( X \)) is a closed set. Where by construction \( N_i \subset D_i \) hence \( N_i \cap D_i = \phi \), and since \( D_i \) and \( D_j \) are disjoint closed sets in the relative topology, we have also \( D_i \cap D_j = \phi \). \( \bar{D}_i \) denotes the closure of \( D_i \) in \( X \), in this paper if \( A \) is a subset of \( X \), then by \( \bar{A} \) we denote the closure of \( A \) in \( X \). Since \( X \) is an open set, we can see \( X \) is a closed set, and hence \( X = X \cap \bar{X} = \phi \). Let \( M = \bar{X}_1 \cap \bar{X}_2 \), then

\[
\begin{align*}
\bar{X}_1 \cap \bar{X}_2 = \bar{X}_1 \cap \bar{X}_2 &= (\bar{N}_1 \cup \bar{D}_1) \cap (\bar{N}_2 \cup \bar{D}_2) \\
&= (\bar{N}_1 \cap \bar{N}_2) \cup (\bar{D}_1 \cap \bar{D}_2) \cup (\bar{D}_1 \cap \bar{N}_2) \cup (\bar{N}_1 \cap \bar{D}_2) \\
&= M \cap \bar{X} - D \\
&= M \cap X - D \\
\end{align*}
\]

Thus if \( M \) is not void then it is a cutting set, and for each \( x \in M \), for all pairs \( (f, g) \in L \), we have \( f(x) = g(x) \). Since \( x \in \bar{X}_i \) and \( x \notin D_i \), implies \( f(x) = g(x) \) for all pair \( (f, g) \in L \), we have \( L_{X_i} \cong L_{D_i} \cong L_1 \) and \( L_{\bar{X}_i} \cong L_{\bar{D}_i} \cong L_2 \). If we recall the \((***) \) part of THEOREM 1, we see that \( M \) must be a void set. And also in a similar method as THEOREM 1, we have \( X_i \) is not empty, since \( L_i \) has two elements. We conclude \( X_i (i = 1, 2) \) are two disjoint open and closed sets. We proved THEOREM 2.

We can see that all of the two characters of which one is used in THEOREM 1, and the other used in THEOREM 2, are not the necessary conditions of the of the PROBLEM needs.

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