◇論　文◇

A NOTE ON THE COMPLETENESS OF UNIFORM SPACES

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§1. Introduction

It is introduced in [2, p. 193] that a uniform space satisfying the first axiom of countability would be complete if every Cauchy sequence converged to a point of the space. But this conjecture has been proved to be false in [1].

It is natural to consider whether the above conjecture holds if the first axiom of countability is replaced by the second axiom of countability.

The main purpose of this note is to show that the answer in general is in the negative. In §3 we will give a counter example for the purpose.

Terminology will be adopted mainly according to [2].

§2. Defitions and theorems.

As a preparation for the following section we shall describe some definitions and theorems which can be found in [2].

Let D be a directed set with binary reltion $\geqq$.

(Definition 1) A net $\{x_n \mid n \in D\}$ is eventually in a set A iff there is an element $m \in D$ such that if $n \in D$ and $n \geqq m$ then $x_n \in A$.

(Definition 2) A net $\{x_n \mid n \in D\}$ in the uniform space (X, $\mathfrak{U}$) is a Cauchy net iff the net $\{(x_m, x_n)\ (m,n) \in D \times D\}$ is eventually in each member of the uniformity $\mathfrak{U}$.

(Defintion 3) A uniform space is complete iff every Cauchy net in the space converges to a point of the space.

(Theorem 1) A family $\mathfrak{S}$ of subsets of $X \times X$ is a subbase for some uniformity for X if

(a) each member of $\mathfrak{S}$ contains the diagonal Δ.

(b) for each U in $\mathfrak{S}$ the set U^{-1} contains a member of $\mathfrak{S}$ and

(c) for each U in $\mathfrak{S}$ there is a V in $\mathfrak{S}$ such that $V \circ V \subset U$.

(Theorem 2) If $\mathfrak{S}$ is a subbase for the uniformity then for each x the family of sets U[x] for U in $\mathfrak{S}$ is a subbase for the neighborhood system of x.

§3. Lemmas and main theorem.

In this section we will construct a uniform space satisfying the second axiom of countability in which every Cauchy sequence converges to a point but some Cauchy net does not converge. Such a space is not complete by Def. 3.

Let X be the set of all rational numbers of the closed unit interval [0, 1].

For each monotone sequence $S = \{x_i \mid i \in \omega\}$ in X, we define a subset V(S) of $X \times X$ such that

$$V(S) = (X - \bigcup_i \{x_i\}) \times (X - \bigcup_i \{x_i\}) \cup \sum_i (x_i, x_i).$$

(Lemma 1) The family $\mathfrak{S} = \{V(S) \mid S$ is a monotone sequence in X$\}$ is a subbase for some uniformity $\mathfrak{U}$ for X.

(Proof) Each member V(S) of $\mathfrak{S}$ has clearly the following properties① (1), (2), and (3).

(1) V(S) contains the diagonal Δ.

(2) $V(S) = V(S)^{-1}$

(3) $V(S) \circ V(S) = V(S)$.

(1), (2) and (3) are the sufficient condition for the family $\mathfrak{S}$ to be a subbase for some uniformity $\mathfrak{U}$ for X. (q.e.d.)

Then we have

(Lemma 2) The uniform space (X, $\mathfrak{U}$) has the

① cf [1] p. 48.

discrete uniform topology.

(Proof) For an arbitrary point x of X there exists a monotone sequence $S=\{x_i | i\in\omega\}$ in X such that $x_i=x$ for each $i\in\omega$. Since V(S) is a member of the subbase $\mathfrak{S}$ for the uniformity $\mathfrak{U}$, $V(S)[x]=\{x\}$ is a neighborhood of x by Theorem 2. (q.e.d.)

(Lemma 3) A sequence $S=\{x_i | i\in\omega\}$ in $(X, \mathfrak{U})$ is a Cauchy sequence iff there is a $k\in\omega$ such that $x_m=x_n$ for every $m,n\geqq k$.

(Proof) Suppose that a sequence $S=\{x_i | i\in\omega\}$ in $(X, \mathfrak{U})$ does not satisfy the specified property in Lemma 3. Then for the sequence S two cases occur as follows:

Case 1. If the sequence S consists of finite members of X, then $\{(x_i, x_j) | (i, j)\in\omega\times\omega\}$ is not eventually in some member $V(S')$ where S' is a monotone sequence which consists of the some members of S.

Case 2. If the sequence S consists of infinite members of X, then we may take a subsequence S'' of S as a monotone sequence. therefore,

$$S\times S=\{(x_i, x_j) | (i, j)\in\omega\times\omega\}$$

is not eventually in $V(S'')$.

Hence for any two cases the sequence S is not a Cauchy sequence.

The converse is obvious. (q.e.d.)

(Lemma 4) Every Cauchy sequence in $(X, \mathfrak{U})$ converges to one point of the space.

(Proof) It is clear by Lemma 3. (q.e.d.)

(Lemma 5) There exists some Cauchy net in $(X, \mathfrak{U})$ which can not converge to a point of the space.

(Proof) For each monotone sequence $S=\{x_i | i\in\omega\}$ in X, let us take a subset $X(S)=X-\bigcup_i \{x_i | x_i\in S\}$ of X.

It is clear that X(S) is a non empty subset of X, and the family $F=\{X(S) | S$ is a monotone sequence in X$\}$ has the finite intersection property. Let $\mathfrak{A}$ be the family of finite intersections of members of F. $\mathfrak{A}$ is directed by $\subset$. Since each member Y_α in $\mathfrak{A}$ is a non empty subset of X we may choose a point y_α in Y_α. Then the net $\{(y_\alpha, y_\beta) | (Y_\alpha, Y_\beta)\in\mathfrak{A}\times\mathfrak{A}\}$ is eventually in each member of the uniformity $\mathfrak{U}$. Because, for an arbitrary member U in $\mathfrak{U}$, there are some members $V(S_1)$, $V(S_2)$, ……, $V(S_n)$ in $\mathfrak{S}$ such that $V(S_1)\cap V(S_2)\cap\cdots\cap V(S_n)\subset U$, and since there is a member $Y_\gamma=X-\bigcup_j \{x_j | x_j\in S_i, i=1,\cdots, n\}$ in $\mathfrak{A}$, (y_α, y_β) belongs to U whenever Y_α, Y_β follow Y_γ.

Since the net $\{(y_\alpha, y_\beta) | (Y_\alpha, Y_\beta)\in\mathfrak{A}\times\mathfrak{A}\}$ is eventually in each member of the uniformity $\mathfrak{U}$, the net $\{y_\alpha | Y_\alpha\in\mathfrak{A}\}$ is a Cauchy net in $(X, \mathfrak{U})$.

It is sufficient to show that the Cauchy net $\{y_\alpha | Y_\alpha\in\mathfrak{A}\}$ can not converge to a point of the space X. For an arbitrary member Y_α in $\mathfrak{A}$ there is a point y_α in Y_α which is a member of the Cauchy net $\{y_\alpha | Y_\alpha\in\mathfrak{A}\}$. Let $Y_\beta=Y_\alpha-y_\alpha$. Then Y_β is a member of $\mathfrak{A}$ and follows Y_α. Therefore there is a member y_β in Y_β which is a member of the Cauchy net, and $y_\alpha\neq y_\beta$. This shows that for every point x of X the Cauchy net is not eventually in $\{x\}$. Since $(X, \mathfrak{U})$ is a discrete space (by Lemma 2), the Cauchy net can not converge to a point of the space X. (q.e.d.)

We, now, have the following result by the above lemmas.

(Main Theorem)

For the uniform space $(X, \mathfrak{U})$ which is constructed as above,

(1) the second axiom of countability is satisfied,

(2) every Cauchy sequence in $(X, \mathfrak{U})$ converges to one point of the space X, and

(3) there is some Cauchy net in $(X, \mathfrak{U})$ which does not converge to a point of the space.

REFERENCES

1. Youngshik Chang, *On The Completeness of Uniform Spaces*, Kyungpook Math. J. Vol. 5, No. 2, p. 47(1963)
2. J.L. Kelley, *General Topology* (1961)

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