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ON THE LIGHT MAPPINGS[1]

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INTRODUCTION

The main object of the present paper is to describe some properties of light mappings of a compact space X into R^n, and the relation between light mappings and dimension in spaces.

In this paper, all spaces are assumed to be completely regular. However in the light of the relation dim X=dim βX[2], we may be able to assume that the space to be considered is compact, as far as the dimension may be concerned. Motivated by the following theorem, we reach to the Theorem 2. 1. and 2. 2. which are not only a generalization of some aspect of the theorem, but also the main result of this paper. For this purpose we interpose CHAPTER I in which we summarize the basic notations and theorems concerning dimension theory which are pertinent to our subsequent disccusion.

Theorem[3]. X is a separable metric space of dimension$\leqq n$ and I_{2n+1} is a $(2n+1)$-dimensional Euclidean cube. Let I^X_{2n+1} denote the space of all continuous mappings of X in I_{2n+1} with metric given the formula:

$$d(f, g)=\sup_{x\in X} \delta[f(x), g(x)],$$

where δ is the metric in I_{2n+1}. Then if X is compact, the set of all homeomorphisms of X into I_{2n+1} constitutes a dense G_δ set in I^X_{2n+1}

CHAPTER I

Definition 1. 1 A mapping $l: X\to Y$ is said to be light provided that, for each $y\in Y$, the inverse $l^{-}(y)$ is totally disconnected.

Theorem 1. 1[4] Let X be a compact space. For any cover[5] U of X, the set

$$G(U)=\{g\in C_n(X) \mid U: \text{envelops } g^{-}(y) \text{ for all } y\in g[X]\}$$

is open in the metric space $C_n(X)$. If, further, dim[6] $X\leqq n$, then G(U) is also dense.

Theorem 1. 2 (BAIRE) The countable intersection of open dense subsets of a complete metric space is dense in the space.

Definition 1. 2 A subring A of $C^*(X)$, the set of bounded continuous functions of X in R, will be called an algebraic subring provided that

(1) all constant functions belong to A

(2) $f^2\in A$ implies $f\in A$.

An algebraic subring that is closed in the metric topology of $C^*(X)$ will be called an analytic subring.

Definition 1. 3 The analytic dimension of $C^*(X)$ — denoted by ad $C^*(X)$—is defined to be the least cardinal m such that every countable

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[2]βX is the Stone-Čech compactification of X (See [1], Chap. 6.)

[3]See [2] p. 56.

[4]See [1] p. 254.

[5]By a cover of X, we shall mean a finite open cover.

[6]The dimension of X — denoted by dimX — is defined to be the least cardinal m such that every basic cover of X has a basic refinement of order at most m.

Remark. We shall denote the set of all continuous mappings from X into R^n by $C_n(X)$. (In special case, $C_1(X)=C(X)$). The set of all bounded functions in $C_n(X)$ is denoted by $C^*_n(X)$.

family in $C^*(X)$ is contained in an analytic subring having a base of power $\leqq m$.

Theorem 1.3[1] Let X be compact. For g[2] $\in C_n(X)$, the analytic subring Ag with base $\{g_1, \cdots, g_n\}$ is precisely the set of all functions that are constant on every component of $g^{\leftarrow}(y)$, for all $y \in g[X]$.

Theorem 1.4 (KATETOV)[3]. The following are equivalent for any completely regular space X.

(1) $\dim X \leqq n$.

(2) $\mathrm{ad}\, C^*(X) \leqq n$ — i.e., every countable subfamily of $C^*(X)$ is contained in an analytic subring having a base of cardinal $\leqq n$.

(3) Every finite subfamily of $C^*(X)$ is contained in an analytic subring having a base of cardinal $\leqq n$.

CHAPTER II

Theorem 2.1 Let X be a compact metric space. If $\dim X \leqq n$, then the set of all light mappings in $C_n(X)$ is a dense G_δ set in the functional space $C_n(X)$.

Proof. Since X is a compact metric space, then there exists a cover U_i of X such that the mesh of U_i is less than $\frac{1}{i}$ for each $i \in N$.

From theorem 1.1 the set $G(U_i)$ is open dense in $C_n(X)$. Therefore $\bigcap_{i \in N} G(U_i)$ is a G_δ and also dense in $C_n(X)$ by theorem 1.2. Since each $G(U_i)$ contains the set of all light mappings in $C_n(X)$, so does $\bigcap_{i \in N} G(U_i)$.

Let g be a function in $\bigcap_{i \in N} G(U_i)$. If $g^{\leftarrow}(y)$ is not totally disconnected in X for an element $y \in R^n$, then there exists a connected set $K \subset g^{\leftarrow}(y)$, which contains more than one point, say, x_1, x_2 be distinct two points in K. There exists $p \in N$ such that $\frac{1}{p} < d(x_1, x_2)$, where d is the metric in X, and hence $g \bar{\in} G(U_p)$. This is a contradiction. Hence the G_δ set, $\bigcap_{i \in N} G(U_i)$, is the set of all light mappings in $C_n(X)$.

Corollary. Let X be a compact metric space. If $\dim X \leqq n$, then the set of all continuous mappings in $C_n(X)$ which are not light mappings is of the first category in $C_n(X)$.

Lemma. Let X be a compact space such that $\dim X \leqq n$. If there exists a light mapping l in $C_n(X)$, then the set of all light mappings in $C_n(X)$ is a dense G_δ set in $C_n(X)$.

Proof. Let $\{V_i\}$ be a family of covers of X such that

$$V_i = \left\{ l^{\leftarrow}[W_i] \mid W_i \in \mathbf{W}_i, \mathbf{W}_i : \text{a cover of } l[X], \text{ such that the mesh of } \mathbf{W}_i < \frac{1}{i} \right\} \text{ for each } i \in N.$$

It is clear that $\bigcap_{i \in N} G(V_i)$ is a dense G_δ set in $C_n(X)$, and contains the set of all light mappings in $C_n(X)$.

Consider any $g \in \bigcap_{i \in N} G(V_i)$. If g is not a light mapping in $C_n(X)$, then there exists $y \in R^n$ such that $g^{\leftarrow}(y)$ contains a nondegenerate connected set $F \subset X$.

By assumption F is contained in a member of V_i for all $i \in N$. We can select two distinct points x_1, x_2 in F such that $l(x_1) \neq l(x_2)$, since l is a light mapping in $C_n(X)$. Hence there exists $p \in N$ such that $\frac{1}{p} < \delta(l(x_1), l(x_2))$, where δ is metric in R^n. Since any member of $\mathbf{W}_p$ can not contain two points $l(x_1)$, $l(x_2)$, then V_p could not envelop F. This is a contradiction.

Theorem 2.2 Let X be a compact space. If there exists a light mapping $g \in C_n(X)$, then the set of all light mappings in $C_n(X)$ is a dense G_δ set in $C_n(X)$.

Proof. By theorem 1.3, the analytic subring A_g with base $\{g_1, \cdots, g_n\}$ is $C(X)$, since g is a light mapping in $C_n(X)$. Hence every countable subfamily of $C(X)$ is contained in A_g. Con-

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[1] See [1], p. 258.

[2] Each n-tuple of functions $g_1, \cdots, g_n \in C(X)$ determine an element $g \in C_n(X)$, defined by:
$g(x) = (g_1(x), \cdots, g_n(x))$ $(x \in X)$.
Conversely, given $g \in C_n(X)$, its coordinate functions $g_i (= \pi_i \circ g)$ belong to $C(X)$.

[3] See [1], p. 259.

$$-id^{ijk}(F^k_\mu F^j_{\mu5} + F^j_{\mu5} F^k_\mu)$$

$$= id^{ijk}(F^k_{\mu5} F^j_\mu + F^j_\mu F^k_{\mu5}) - id^{ijk}(F^k_{\mu5} F^j_\mu + F^j_\mu F^k_{\mu5}) \quad \therefore [F^i_5, F^j_\mu F^j_\mu - F^j_{\mu5} F^j_{\mu5}] = 0.$$

Next using $F^i_\lambda F^k_\nu = F^i_{\lambda5} F^k_{\nu5}$ and also

$$F^i_\lambda F^k_{\kappa5} = -F^i_{\lambda5} F^k_\kappa, \text{ since } F^k_{\lambda5} F^k_{\nu5} = \gamma_\lambda \gamma_\nu T^i T^k,$$

$$[F^i_\lambda F^i_\lambda - F^i_{\lambda5}, F^i_{\lambda5} F^j_{\mu\nu}] = 0,$$

Completely similar manner, but after much more lengthy calculation, we can prove that F^i, F^i_5, $F^j_{\mu\nu}$ commute with

$$F^j F^j - F^j_5 F^j_5 + \frac{1}{2} F^j_{\mu\nu} F^j_{\mu\nu}.$$

Notice, since we have a complete table for Salam algebra, the proof is no more difficult than previous one, only now we have more factors to take care of, therefore it is lengthy, we omit this proof only because of space limitations.

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sidering X is compact, and hence $C^*(X) = C(X)$, ad $C(X) \leqq n$ by Theorem 1.4. Hence $\dim X \leqq n$. By the previous lemma, the set of all light mappings in $C_n(X)$ is a dense G_δ set in $C_n(X)$.

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