Ũ(12) ALGEBRA

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1. INTRODUCTION

The problem of finding a relatvistic generaliztion of the U(6) group in connection with the internal symmetry of elementary particles has engaged considerable attention recently. In particular, Salam's paper (1965, Ref. I) gave a great stimulus to study the Ũ(12) group structure. It is the purpose of this paper to give a comprehensive list of formulas and commutators of Ũ(12) algebra in order to assist physicists for further study of the most important and central problem of physics today. Rather complicated computational techniques are involved, therefore we give several examples and proofs of formulas which are given without proofs in Ref. I (incidently correcting mistakes (or misprints) of formulas are given in Ref. I).

2. Ũ(12) ALGEBRA and SUBALGEBRA

The group structure Ũ(12) is defined by the algebra of the 144 matrices

$$F^{Ri}=\gamma^R T^i,\quad R=1,\cdots,16;\ i=0,\cdots,8.$$

Here $\gamma^R=I,\gamma_\mu,\sigma_{\mu\nu}=\frac{1}{2}i[\gamma_\mu,\gamma_\nu],$

$\sigma_{\mu5}=i\gamma_\mu\gamma_5,\ \gamma_5\ (\mu=0,1,2,3)$

are the well-known 16 matrices of Dirac algebra, which satisfy the relations

$$[\gamma_\mu,\gamma_\nu]=\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2g_{\mu\nu}\ (\mu,\nu=0,1,2,3)$$

with γ^0 hermitian and γ antihermitian and the signature of the metric $(1,-1,-1,-1)$. That is,

$$g_{00}=-g_{11}=-g_{22}=-g_{33}=1,$$

$$g_{\mu\nu}=0 \text{ if } \mu\neq\nu,$$

$$\gamma_0^+=\gamma,\ \gamma_1^+=-\gamma_1,\ \gamma_2^+=-\gamma_2,\ \textit{and}\ \gamma_3^+=-\gamma_3$$

$T^i=\frac{1}{2}\lambda^i$ are the well-known 9 unitary spin matrices, with the λ^i defined by Gell-Mann(1962, Ref. 2).

Thus $\mathrm{Tr}(T^iT^j)=\frac{1}{2}\delta^{ij},\ f^{0jk}=0,$

$$d^{0jk}=\delta^{jk}(2/3)^{\frac{1}{2}}$$

$$\left.\begin{aligned}&[T^i,T^j]=if^{ijk}T^k\\&\{T^i,T^j\}=T^iT^j+T^jT^i=d^{ijk}T^k\end{aligned}\right\}\quad(1)$$

where the representation of f^{ijk}, d^{ijk} are given in Ref. 2.

Since γ^R characterizes the space-time symmetry (the representations of Poincare group or inhomogeneous Lorentz group) and T^i characterizes the internal symmetry of elementary particles, $\gamma^R(R=1,2,\cdots,16)$ and $T^i(i=0,1,\cdots,8)$ commute each other.

Therefore in order to know the Lie algebraic characters of Ũ(12) group, i.e. the closure of all Lie products (commutators) of 144 elements, we must compute

$$[\gamma^AT^i,\gamma^BT^j],\ (A,B=1,\cdots16,i,j=0,\cdots,8).$$

Fundamental Identities

$$[\gamma^AT^i,\gamma^BT^j]=\frac{1}{2}\{\gamma^A,\gamma^B\}[T^i,T^j]+\frac{1}{2}[\gamma^A,\gamma^B]\{T^i,T^j\}$$

which is easily checked by remembering { } as anticommutator and [ ] commutator brackets.

By (1) we obtain immediately

$$[\gamma^AT^i,\gamma^BT^j]=\frac{1}{2}i\{\gamma^A,\gamma^B\}f^{ijk}T^k+\frac{1}{2}[\gamma^A,\gamma^B]d^{ijk}T^k\quad(2)$$

By (2) it is clear that we have to know all commutators and anticommutators of the Dirac algebra (i.e. γ^A), and it is important, but enough they are closed under Lie products i.e. all commutators (Lie products of any 2 elements of γ^A) must produce nothing but a linear combination of γ^A themselves, since by (1) we already know explicitely all commutators and anticommutators of T^i (i.e. Gell-Mann algebra SU(3) is closed).

We give the complete listing of all commutators and anticommutators of Dirac algebra. Proofs of some difficult ones are given in § 3.

$$[1, \sigma_{\mu\nu}]=0, \quad \{1, \sigma_{\mu\nu}\}=2\sigma_{\mu\nu},$$
$$[\gamma_\lambda, \sigma_{\mu\nu}]=2i(g_{\lambda\mu}\gamma_\nu-g_{\lambda\nu}\gamma_\mu),$$
$$\{\gamma_\lambda, \sigma_{\mu\nu}\}=-2\varepsilon_{\lambda\mu\nu\rho}\sigma_{\rho5})$$
$$[\sigma_{\kappa\lambda}, \sigma_{\mu\nu}]=2i(g_{\kappa\nu}\sigma_{\lambda\mu}+g_{\lambda\mu}\sigma_{\kappa\nu}$$
$$-g_{\kappa\mu}\sigma_{\lambda\mu}-g_{\lambda\nu}\sigma_{\kappa\mu}),$$
$$\{\sigma_{\kappa\lambda}, \sigma_{\mu\nu}\}=2(g_{\kappa\mu}g_{\lambda\nu}-g_{\lambda\mu}g_{\kappa\nu})-2\varepsilon_{\kappa\lambda\mu\nu}\gamma_5,$$
$$[\sigma_{\lambda5}, \sigma_{\mu\nu}]=2i(g_{\lambda\mu}\sigma_{\lambda5}-g_{\lambda\nu}\sigma_{\mu5})$$
$$\{\sigma_{\lambda5}, \sigma_{\mu\nu}\}=-2\varepsilon_{\lambda\mu\nu\rho}\gamma_\rho$$
$$[\gamma_\mu, \sigma_{\nu5}]=2ig_{\mu\nu}\gamma_5, \quad \{\gamma_\mu, \sigma_{\nu5}\}=\varepsilon_{\mu\nu\rho\sigma}\sigma_{\rho5}$$
$$[\gamma_5, \sigma_{\mu\nu}]=0, \quad \{\gamma_5, \sigma_{\mu\nu}\}=\varepsilon_{\mu\nu\rho\lambda}\sigma_{\rho\lambda}.$$

With the above formulas and (1), (2), we can now compute $[F^{Ri}, F^{Rj}]$ remembering $F^{Ri}\equiv\gamma^R T^i$. Using the notation with $F^i\equiv IT^i, F_5^i\equiv\gamma_5 T^i$

$$F_{\mu\nu}^j\equiv\sigma_{\mu\nu}T^j, \quad F_\mu^i\equiv\gamma_\mu T^j, \quad F_{\lambda5}^j=\sigma_{\lambda5}T^j,$$

we deduce the following commutators:

W(6)
$$[F^i, F^j]=if^{ijk}F^k, \quad [F^i, F_5^j]=if^{ijk}F_5^k,$$
$$[F_5^i, F_5^j]=-if^{ijk}F^k, [F^i,F_{\mu\nu}^j]=if^{ijk}F_{\mu\nu}^k,$$
$$[F_5^i, F_{\mu\nu}^j]=\frac{1}{2}if^{ijk}\varepsilon_{\mu\nu\kappa\lambda}F_{\kappa\lambda}^k.$$
$$[F_{\kappa\lambda}^i, F_{\mu\nu}^j]=id^{ijk}(g_{\kappa\nu}F_{\lambda\mu}^k+g_{\lambda\mu}F_{\kappa\nu}^k-g_{\kappa\mu}F_{\lambda\nu}^k$$
$$-g_{\lambda\nu}F_{\kappa\nu}^k)+if^{ijk}\{(g_{\kappa\mu}g_{\lambda\nu}-g_{\lambda\mu}g_{\kappa\nu})F^k$$
$$-\varepsilon_{\kappa\lambda\mu\nu}F_5^k\},$$

$$[F_\mu^i, F_\nu^j]=if^{ijk}g_{\kappa\nu}F^k-id^{ijk}F_{\mu\nu}^k,$$
$$[F_\mu^i, F_{\nu5}^j]=id^{ijk}g_{\mu\nu}F_5^k+\frac{1}{2}if^{ijk}\varepsilon_{\mu\nu\kappa\lambda}F_{\kappa\lambda}^k,$$
$$[F_{\mu5}^i, F_{\nu5}^j]=-if^{ijk}g_{\mu\nu}F^k-id^{ijk}F_{\mu\nu}^k,$$
$$[F^i, F_\mu^j]=if^{ijk}F_\mu^k, \quad [F^i, F_{\mu5}^j]=if^{ijk}F_{\mu5}^k$$
$$[F_5^i, F_\mu^j]=id^{ijk}F_{\mu5}^k, \quad [F_5^i, F_{\mu5}^j]=id^{ijk}F_\mu^k,$$
$$[F_\lambda^i, F_{\mu\nu}^j]=id^{ijk}(g_{\lambda\mu}F_\nu^k-g_{\lambda\nu}F_\mu^k)$$
$$-if^{ijk}\varepsilon_{\lambda\mu\nu k}F_{k5}^k,$$
$$[F_{\lambda5}^i, F_{\mu\nu}^j]=id^{ijk}(g_{\lambda\mu}F_{\nu5}^k-g_{\lambda\nu}F_{\mu5}^k)$$
$$+if^{ijk}\varepsilon_{\lambda\mu\nu k}F_k^k.$$

From the above commutation rule for F^{Ri} it is clear that every Lie products of any two of F^{Ri} produce nothing but a linear combination of F^{Ri} themselves. Since it is closed under multiplication, it is algebra. Let us call it Salam algebra. This is a special case of Lie algebra. As it is well known, γ's has 4-dimensional representation, and T^i are generated by 3 fundamental quarks(the famous eightfold way of Gell-Mann). Therefore F^{Ri} (Salam algebra) are the algebra of 144 linearly independent matrices whose representation are clearly ($12\times12=144$) a minimum 12 dimensional operating on a 12-components Dirac quark(multispinor). Following the procedure of Lie algebra we find that the quadratic Casimir operator is

$$F^jF^j-F_5^jF_5^j+\frac{1}{2}F_{\mu\nu}^jF_{\mu\nu}^j+F_\mu^jF_\mu^j-F_{\mu5}^jF_{\mu5}^j.$$

Insepection of the commutators for $\tilde{U}(12)$ reveals that a 72-component of subaglebra is genenated by the operators F^j, F_5^j, $F_{\mu\nu}^j$.

That is: every commutators of any two of F^j, F_5^j, $F_{\mu\nu}^j$ are closed under Lie product. This is the subgroup W(6) and in the fundamental representation has the generators T^j, γ_5T^j and $\sigma_{\mu\nu}T^j$. The expressions $F_\mu^jF_\mu^j$ $F_{\mu5}^jF_{\mu5}^j$ and $F^jF^j-F_5^jF_5^j+\frac{1}{2}F_{\mu\nu}^jF_{\mu\nu}^j$ are now separately invariant under W(6). Proof is given at § 3. Note that W(6) posseses also the important 36-parameter subgroup U(6) ($T^i\sigma_{ab};T^i$; $a,b=1,2,3$). This is identcal with U(6) of Gursey, Radicati and Sakita.

Therefore $\tilde{U}(12)$ contains automatically in itself U(6) subgroup and finally SU(3) subgroup of Gell-Mann. It is this fact that makes its importance in the theory of elementary particles paramount. The full symmetry(although this is broken badly) of nature is contained in $\tilde{U}(12)$.

3. MATHEMATICS of $\tilde{U}(12)$

Since metric is given as signature(+, −, −, −)

$$\gamma^\mu=g^{\mu\nu}\gamma_\nu, \quad \gamma_0^2=1, \quad \gamma_1^2=\gamma_2^2=\gamma_3^2=-1$$

follow from the commutation relations $[\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu]=2g_{\mu\nu}$, putting $\nu=\mu=0,1,2,3$. Space part ($\mu=1,2,3$) indices can be lowered or raised only by change of sign, (i. e. multiplied by −1). But the time component ($\mu=0$) of covariant and contravariant tensor are the same.

$$-\gamma^5=\gamma_5\equiv\gamma_0\gamma_1\gamma_2\gamma_3 \quad \therefore \ (\gamma^5)^2=-1$$

also $\begin{cases}\gamma^5\gamma^\mu+\gamma^\mu\gamma^5=0\\ \gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma=\varepsilon_{\mu\nu\rho\sigma}\gamma_5 \ (\mu\neq\nu\neq\rho\neq\sigma).\end{cases}$

Among cmmutators and auticommutators of Dirac algebra we give only the following proofs due to space limitations. The rest can be easily checked.

THEOREM 1.

$$[\sigma_{\kappa\lambda},\sigma_{\mu\nu}]=2i(g_{\kappa\nu}\sigma_{\lambda\mu}+g_{\lambda\mu}\sigma_{\kappa\nu}-g_{\kappa\mu}\sigma_{\lambda\mu}-g_{\lambda\mu}\sigma_{\kappa\mu}).$$

PFOOF.

$$[\sigma_{\kappa\lambda},\sigma_{\mu\nu}]=[i\gamma_\kappa\gamma_\kappa,\sigma_{\mu\nu}]$$
$$=i(\gamma_\kappa\gamma_\lambda\sigma_{\mu\nu}-\gamma_\kappa\sigma_{\mu\nu}\gamma_\lambda+\gamma_\kappa\sigma_{\mu\nu}\gamma_\lambda-\sigma_{\mu\nu}\gamma_\kappa\gamma_\lambda)$$
$$=i\{\gamma_\kappa[\gamma_\lambda,\sigma_{\mu\nu}]+[\gamma_\kappa,\sigma_{\mu\nu}]\gamma_\kappa\}.$$

Using $[\gamma_\lambda,\sigma_{\mu\nu}]=2i(g_{\lambda\mu}\gamma_\nu-g_{\lambda\nu}\gamma_\mu)$

$$=i[\gamma_\kappa 2i(g_{\lambda\mu}\gamma_\nu-g_{\lambda\nu}\gamma_\mu)+2i(g_{\kappa\mu}\gamma_\nu-g_{\kappa\nu}\gamma_\mu)\gamma_\lambda]$$
$$=2ii[g_{\lambda\mu}\gamma_\kappa\gamma_\nu-g_{\lambda\nu}\gamma_\kappa\gamma_\mu+g_{\kappa\mu}\gamma_\nu\gamma_\lambda-g_{\kappa\nu}\gamma_\mu\gamma_\lambda]$$
$$=2i(g_{\kappa\nu}\sigma_{\lambda\mu}+g_{\lambda\mu}\sigma_{\kappa\nu}-g_{\kappa\mu}\sigma_{\lambda\nu}-g_{\lambda\nu}\sigma_{\kappa\mu})$$

THEOREM 2.

$$\{\sigma_{\kappa\lambda},\sigma_{\mu\nu}\}=2(g_{\kappa\mu}g_{\lambda\nu}-g_{\lambda\mu}g_{\kappa\nu})-2\varepsilon_{\kappa\lambda\mu\nu}\gamma_5.$$

PROOF.

$$\{\sigma_{\kappa\lambda},\sigma_{\mu\nu}\}=\{i\gamma_\kappa\gamma_\lambda, i\gamma_\mu\gamma_\nu\}.$$

But

$$\gamma_\mu\gamma_\nu\gamma_\kappa\gamma_\lambda=\gamma_\mu(2g_{\nu\kappa}-\gamma_\kappa\gamma_\nu)\gamma_\lambda=2g_{\nu\kappa}\gamma_\mu\gamma_\lambda-\gamma_\mu\gamma_\kappa\gamma_\nu\gamma_\lambda$$
$$=2(g_{\nu\kappa}\gamma_\mu\gamma_\lambda-g_{\mu\kappa}\gamma_\nu\gamma_\lambda+g_{\nu\lambda}\gamma_\kappa\gamma_\mu-g_{\mu\lambda}\gamma_\kappa\gamma_\nu)$$
$$+\gamma_\kappa\gamma_\lambda\gamma_\mu\gamma_\nu.$$

But

$$\gamma_\kappa\gamma_\lambda\gamma_\mu\gamma_\nu=(-g_{\kappa\mu}\gamma_\lambda\gamma_\nu+g_{\kappa\nu}\gamma_\lambda\gamma_\mu+g_{\lambda\mu}\gamma_\kappa\gamma_\nu-g_{\lambda\nu}\gamma_\kappa\gamma_\mu)$$
$$+(g_{\lambda\mu}g_{\kappa\nu}-g_{\kappa\mu})+\varepsilon_{\kappa\lambda\mu\nu}\gamma_5.$$

In arriving the above identity, knowing $\kappa\neq\lambda$ and $\mu\neq\nu$, we exhaust all possible cases by enumerating that indices can be 1) $\kappa=\mu$, 2) $\kappa=\nu$, 3) $\lambda=\mu$, 4) $\lambda=\nu$; 5) $\kappa=\nu$ and $\lambda=\nu$, 6) $\kappa=\nu$ and $\lambda=\mu$, 7) $\kappa\neq\lambda\neq\mu\neq\nu$.

$$\therefore \ \{\sigma_{\kappa\lambda},\sigma_{\mu\nu}\}=2(g_{\kappa\mu}g_{\lambda\nu}-g_{\lambda\mu}g_{\kappa\nu})-2\varepsilon_{\kappa\lambda\mu\nu}\gamma_5.$$

Knowing the complete tables of commutators and anticommutators of Dirac algedra, it is now trivial matter to compute any Lie product of Salam algebra.

By the fundamental identity, it follows that

$$[F^i_{\kappa\lambda},F^j_{\mu\nu}]=[\sigma_{\kappa\lambda}T^i,\sigma_{\mu\nu}T^j]=\frac{1}{2}[\sigma_{\kappa\lambda},\sigma_{\mu\nu}]d^{ijk}T^k$$
$$+\frac{1}{2}\{\sigma_{\kappa\lambda},\sigma_{\mu\nu}\}if^{ijk}T^k$$

using the above derived formulas (Theorem 1, and Theorem 2), we have immediately

$$=id^{ijk}(g_{\kappa\nu}\sigma_{\lambda\mu}+g_{\lambda\mu}\sigma_{\kappa\nu}-g_{\kappa\mu}\sigma_{\lambda\nu}-g_{\lambda\nu}\sigma_{\kappa\mu})T^k$$
$$+if^{ijk}\{g_{\kappa\mu}g_{\lambda\nu}-g_{\lambda\mu}g_{\kappa\nu})-\varepsilon_{\kappa\lambda\mu\nu}\gamma_5\}T^k.$$

Next we turn our attention to sublgebra W(6) of $\tilde{U}$ (12). It is clear from the table of multiplication of Salam algebra that a 72 component subalgebra is generated by the elements F^j, $F_5{}^j$, $F_\mu{}^j{}_\nu$ (i. e. $j=0,\cdots,8$, $\mu\neq\nu=0,1,2,3$ $\quad\therefore\ 9+9+9\times6=72$). By the general theory of Lie algebra about the quadratic Casmir operator (Ref. 3). the following two expressions are now separately invariant uncer W(6) i. e. they commute with the generators F^j, $F_5{}^j$, $F^j_{\mu\nu}$ of W(6) algebra, therefore also commute each other.

THEOREM 3.

The expressions

$F^j_\mu F^j_\mu-F^j_{\mu5}F^j_{\mu5}$ and $F^jF^j-F^j_5F^j_5+\frac{1}{2}F^j_{\mu\nu}F^j_{\mu\nu}$ commute separately with each generator F^j, F^j_5, $F^j_{\mu\nu}$ of W(6), therefore are invariant of W(6).

PROOF: Notice the following identities .

[A, BB]=[A, B]B+B[A, B]

[AA, B]=A[A, B]+[A, B]A.

Using the multiplication table of Salam algebra, remembering that f^{ijk} are completely antisymmetric with all indices i, j, k and d^{ijk} are completely symmetric with all indices i, j, k,

$$[F^i, F^j_\mu F^j_\mu]=[F^i, F^j_\mu]F^j_\mu+F^j_\mu[F^i, F^j_\mu]$$
$$=if^{ijk}(F^k_\mu F^j_\mu+F^j_\mu F^k_\mu)=0$$

since $[F^i, F^j_\mu]=if^{ijk}F^k_\mu$, $f_{ijk}=-f_{ikj}$, $F^k_\mu F^j_\mu+F^j_\mu F^k_\mu$ is symmetric with respect to indices k, i.

By the completely analogous manner,

$$[F^i, F^j_{\mu5}F^j_{\mu5}]=0.\ [F^i, F^j_\mu F^j_\mu-F^j_{\mu5}F^j_{\mu5}]=0.$$

Similarly, using

$$[F^i_5, F^j_\mu]=id^{ijk}F^k_{\mu5},$$
$$[F^i_5, F^j_{\mu5}]=id^{ijk}F^k_\mu,\quad d^{ijk}=d^{ikj},$$
$$[F^i_5, F^j_\mu F^j_\mu-F^j_{\mu5}F^j_{\mu5}]=id^{ijk}(F^k_{\mu5}F^j_\mu+F^j_\mu F^k_{\mu5})$$

$$-id^{ijk}(F^k_\mu F^j_{\mu 5} + F^j_{\mu 6} F^k_\mu)$$

$$= id^{ijk}(F^k_{\mu 5} F^j_\mu + F^j_\mu F^k_{\mu 5}) - id^{ijk}(F^k_{\mu 5} F^j_\mu$$

$$+ F^j_\mu F^k_{\mu 5}) \quad \therefore [F^i_5, F^j_\mu F^j_\mu - F^j_{\mu 5} F^j_{\mu 5}] = 0.$$

Next using $F^i_\lambda F^k_\nu = F^i_{\lambda 5} F^k_{\nu 5}$ and also

$F^i_\lambda F^k_{\kappa 5} = - F^i_{\lambda 5} F^k_k$, since $F^k_{\lambda 5} F^k_{\nu 5} = \gamma_\lambda \gamma_\nu T^i T^k$,

$[F^i_\lambda F^i_\lambda - F^i_{\lambda 5}, F^i_{\lambda 5} F^j_{\mu\nu}] = 0$,

Completely similar manner, but after much more lengthy calculation, we can prove that F^i, F^j_5, $F^j_{\mu\nu}$ commute with

$$F^j F^j - F^j_5 F^j_5 + \frac{1}{2} F^j_{\mu\nu} F^j_{\mu\nu}.$$

Notice, since we have a complete table for Salam algebra, the proof is no more difficult than previous one, only now we have more factors to take care of, therefore it is lengthy, we omit this proof only because of space limitations.

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sidering X is compact, and hence $C^*(X) = C(X)$, ad $C(X) \leqq n$ by Theorem 1.4. Hence $\dim X \leqq n$. By the previous lemma, the set of all light mappings in $C_n(X)$ is a dense G_δ set in $C_n(X)$.

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