The Simple Model of the Recurrence Time
in the Reproductive Pattern of a Married Female

CHI-HOON CHOI

0. Introduction

This paper is to treat entirely through the Bayesian Statistics. The recurrence time has been studied by many scholars but this shall be the first trial to approach by the subjective probabilistic idea.

Biological considerations make it natural to view as a random variable the length of time spent in a state after the beginning of marriage or after return from another state.

The model which is used in this paper is very simple one. We may introduce more complicated states such as not only a live-birth but also still-birth and induced or spontaneous abortion. However, the Bayesian approach can also be handled with stochastic process for this problem.

This paper does not include how to decide the parameters which appear in prior distribution and how to collect data to fit formulas.

1. The model

One sets up the model of the reproductive pattern of a married female under the assumption that a woman is in one and only one of the following states:

\[ s_0 = \text{nonpregnant and fecundable}, \]
\[ s_1 = \text{pregnant state to be terminated in livebirth}, \]
\[ s_2 = \text{postpartum sterile period associated with only livebirth}. \]

We shall assume that a female in the state \( s_0 \) is subject at intervals of approximately one month to a probability \( \tilde{p}_1 \) of entering state \( s_1 \), that the female in the state \( s_1 \) has the probability 1 of entering state \( s_2 \), and that the female in the state \( s_2 \) is subject to a probability \( \tilde{p}_2 \) of entering state \( s_0 \).

Then the total number of months \( t_1 \) spent in state \( s_0 \) during any single visit to that state has a density

\[
g_1(t_1|\tilde{p}_1) = \tilde{p}_1(1-\tilde{p}_1)^{t_1-1}, \quad t_1 = 1, 2, \ldots; \quad 0 \leq \tilde{p}_1 \leq 1.
\]

Similarly we have for state \( s_2 \)

\[
g_2(t_2|\tilde{p}_2) = \tilde{p}_2(1-\tilde{p}_2)^{t_2-1}, \quad t_2 = 1, 2, \ldots; \quad 0 \leq \tilde{p}_2 \leq 1.
\]

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where \( t_2 \) is the total number of months spent in state \( s_2 \) during a single visit to that state.

But in state \( s_1 \), the time \( t \) spent in that state is to be assumed that \( t' \) is distributed by a Normal density with a known and an unknown \( \sigma^2 \).

Now we can see very simple reproductive process in which a woman, beginning in state \( s_0 \), passes after a period of time into state \( s_1 \), thence, with a probability \( s_2 \) and back to state \( s_0 \), to begin the reproductive cycle again. This is a Renewal Process.

2. The likelihood and sufficient statistics

(i) For the states \( s_0 \) and \( s_2 \)

The likelihood of \( t_{1i} \) is

\[
g_1(t_{1i} \mid p_i) = p_i(1-p_i)^{n_i-1}
\]

thus the likelihood of observations \( t_{11}, \ldots, t_{1n} \) which are independently and identically distributed (i.i.d) by \( t_{1i} \sim p_i(1-p_i)^{n_i-1} \), given \( p_i \) is given by

\[
g_1(t_{11}, \ldots, t_{1n} \mid p_i) \propto \prod_{i=1}^{n_i} p_i(1-p_i)^{n_i-1} = p_i^s(1-p_i)^{\frac{n-m}{2}}.
\]

Therefore \((n, \sum t_{1i})\) is sufficient for \( p_i \).

(ii) For state \( s_1 \)

The likelihood of \( t' \) is

\[
g(t', \ldots, t'_{n_i} \mid \mu, \sigma) \propto \frac{1}{\sigma^m} \exp \left\{ -\frac{\sum (t'_{1i} - \mu)^2}{2\sigma^2} \right\} \propto \frac{1}{\sigma^m} \exp \left\{ -\frac{ms^2}{2\sigma^2} \right\}
\]

where \( s^2 = \frac{\sum (t'_{1i} - \mu)^2}{m} \) and \( m \) is a number of observations which are i.i.d a Normal with a mean \( \mu \) and a variance \( \sigma^2 \).

Thus the sufficient statistics are \( m \) and \( s^2 \) for \( \sigma^2 \) when \( \mu \) is known.

3. The conjugate prior and the posterior distribution

(i) For the states \( s_0 \) and \( s_2 \)

(a) The conjugate prior

The conjugate prior for \( p_i \) is appropriate if it is Beta density, that is,

\[
g_1(p_i \mid v_0, v_1) = k \cdot p_i^{(v_0-1)}(1-p_i)^{(v_1-1)}, \quad 0 \leq p_i \leq 1; \quad v_1 > v_0 > 0,
\]

where \( k = \frac{(v_1-1)!}{(v_0-1)!v_1} \) and \( v_0, v_1 \) are determined by prior knowledges such that

\[
\mu_1 = v_0/v_1, \quad \mu_2 = \frac{v_0(v_1 - v_0)}{v_1(\mu_1 + 1)}
\]
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(b) The posterior density of $\tilde{p}_1$

The posterior density of $\tilde{p}_1$ is proportional to the product of the likelihood and the prior, i.e.,

$$g''(p_1|t_{11}, t_{12}, \ldots, t_{1m}) \propto p_1^*(1-p_1)^{t_{11}+\cdots+t_{1m}-1} \cdot p_1^{\nu-1} (1-p_1)^{\nu-1}.$$  

The standardized constant will be obtained by the integration

$$\int_0^1 p_1^{n+\nu-1} (1-p_1)^{n+\nu-1} dp_1 = \frac{(n+\nu-1)!}{(n_1+n+1-1)!}.$$  

where

$$t_1 = \frac{1}{n} \sum_i t_{ii}.$$  

Hence the posterior density of $\tilde{p}_1$ given the data and $v_0, v_1$ is

$$g''(p_1|\text{data, } v_0, v_1) = \frac{(n_1+v_1-1)!}{(n+v_0-1)!} \cdot \frac{(n_1-n+v_1-v_0-1)!}{(n_1+v_1-1)!}.$$  

$$0 \leq p_1 \leq 1; v_1 > v_0 > 0.$$  

(c) Unconditional density of $t_1$  

The unconditional density of $t_1$ can be obtained by

$$g_1(t_1|v_0, v_1) = \int_0^1 g_1(t_1|p_1) g''(p_1|v_0, v_1) dp_1$$  

$$= \int_0^1 \frac{(n_1+v_1-1)!}{(n+v_0-1)!} \cdot \frac{(n_1-n+v_1-v_0-1)!}{(n_1+v_1-1)!} \cdot \frac{(n_1-v_1-1)!}{(n_1+v_1-1)!}.$$  

Similarly we can obtain that

$$g_1(t_1|v_0, v_1; n) = \frac{(v_1-1)(v_0+n-1)!}{(n_1+v_1-v_0-1)!}.$$  

(ii) For the state $s_1$

(a) Conjugate prior

When the mean $\mu$ of an independent normal process is known but the precision $\sigma^2$ is treated as a random variable, the natural conjugate of

$$\frac{1}{\sigma^2} \exp \left\{ -\frac{m^2}{2\sigma^2} \right\}$$

is

$$g'(\sigma^2|v_0, v_1) = K\sigma^{-(\kappa+1)} \exp \left\{ -\frac{v_0\sigma_1^2}{2\sigma^2} \right\}.$$
where \( K = \frac{(v_0v_1)^{\nu/2}}{2^{\nu-1}} \Gamma \left( \frac{v_0}{2} \right) \) this is known as the Gamma-2 distribution.

(b) Unconditional distribution of \( t' \)

\[
\begin{align*}
g(t'|v_0, v_1, \mu) &= \int_0^\infty g(t'|\mu, \sigma^2)g(\sigma^2|v_0, v_1)d\sigma^2 \\
&= \int_0^\infty \frac{K}{\sqrt{2\pi} \sigma} e^{-\frac{(t'-\mu)^2}{2\sigma^2}} \cdot \frac{1}{\sigma^{v_0+1}} e^{-\frac{v_0\sigma^2}{2\nu}} d\sigma^2 \\
&= \left[1 + \frac{(t'-\mu)^2}{v_0v_1} \right]^{-\frac{v_0+v_1}{2}}.
\end{align*}
\]

Thus \( \frac{t'-\mu}{\sqrt{v_1}} \) has \( t \)-distribution with \( v_0 \) degrees of freedom, i.e., if we put \( T = \frac{t'-\mu}{\sqrt{v_1}} \) then

\[
\begin{align*}
g(T|v_0, v_1, \mu) &= \frac{\Gamma \left( \frac{v_0+1}{2} \right)}{\sqrt{\pi} v_0 \Gamma \left( \frac{v_0}{2} \right)} \frac{1}{\left[1 + \frac{T^2}{v_0}\right]^{\frac{v_0+v_1}{2}}}.
\end{align*}
\]

4. The expectation of \( t \) and \( t' \)

(i) \( t_1 \) has the density

\[
\begin{align*}
g_1(t_1|v_0, v_1) &= \frac{(v_1-1)!v_0!(t_1+v_1-v_0-2)!}{(v_0-1)!(v_1-v_0-1)!(t_1+v_1-1)!}
\end{align*}
\]

\[
E(t_1) = \sum_{i=1}^{\infty} \int_0^\infty t_1 \frac{(v_1-1)!}{(v_0-1)!(v_1-v_0-1)!(t_1+v_1-1)!} \frac{p_1^v (1-p_1)^{v_1-v_0-2} d p_1}{p_1^v (1-p_1)^{v_1-v_0-2} \sum_{i=1}^{\infty} t_1 p_1 (1-p_1)^{v_1-v_0-2} d p_1}
\]

\[
\begin{align*}
&= \int_0^\infty \frac{(v_1-1)!}{(v_0-1)!(v_1-v_0-1)!} \frac{p_1^{v_1-1} (1-p_1)^{v_1-v_0-2} d p_1}{p_1^{v_1-1} (1-p_1)^{v_1-v_0-2}}
&= \frac{v_1-1}{v_0-1}.
\end{align*}
\]

(ii) \( T = \frac{t'-\mu}{\sqrt{v_1}} \) has the density of Student. Hence \( E(T) = 0 \), that is \( E(t') = \mu \).

5. Recurrence time

We can see that \( t_1, t', t_2 \) are independent. Therefore the likelihood of \( t_1, t', t_2 \)
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The unconditional joint density of $t_1, t_1', t_2$ should be

$$f(t_1, t_1', t_2 | \nu_0, \nu_1, \nu_2, \nu_3) \propto p_1^{t_1 + 1}(1 - p_1)^{\nu_0 - t_1 - 1} p_2^{t_1' + 1}(1 - p_2)^{\nu_1 - t_1' - 1}.$$ 

Hence unconditional joint density of $t_1, t_1', t_2$ is

$$f(t_1, t_1', t_2 | \nu_0, \nu_1, \nu_2, \nu_3, \nu_4) \propto p_1^{t_1 + 1}(1 - p_1)^{\nu_0 - t_1 - 1} p_2^{t_1' + 1}(1 - p_2)^{\nu_1 - t_1' - 1}. $$

Let $t_1 + t_1' + t_2 = t$ and let $R$ denote the set of all $t_1, t_1', t_2$ which satisfy $t_1 + t_1' + t_2 \leq t$, then the probability that the recurrence time is less than or equal to a given $t_0$ is given by

$$P(t_1 + t_1' + t_2 \leq t_0) = \sum_{t_1} \sum_{t_1'} f(t_1, t_1', t_2 | \nu_0, \nu_1, \nu_2, \nu_3),$$

$$t_1, t_1', t_2 \in R.$$ 

6. The assumptions on $p_1, \sigma^2, p_2$

$p_1$ is called by fecundability in common The fecundability has been estimated by quite a few scholars under various models. Typical models are as follows:

(i) The model based on the length of the fertile period and the frequency of coitus per menstrual cycle.

(ii) The model using of the total number of children born to couples over 45 in a population reputed to completely avoid contraception.

(iii) The model fitting a theoretical distribution to experience respecting first conception delays.

We have been considering third model, thus the assumption on the fecundability $p_1$ of each couple has to remain $p_1$ constant from month to month until conception.

If $p_1$ or $p_2$ are not identical for each couple the likelihood of $t_1$, will be

$$g(t_1 | p_1) = p_1(1 - p_1)^{t_1 - 1}$$

thus for $t_{11}, t_{12}, \ldots, t_{1n}$

$$g(t_{11}, t_{12}, \ldots, t_{1n} | p_{11}, \ldots, p_{1n}) \propto \prod_{i=1}^{n} p_{1i}(1 - p_{1i})^{t_{1i} - 1}.$$ 

This can be written as
Note that in fact there may not be all different \( p_1, \ldots, p_{m} \), but still we may say that, therefore, \( t_1, \ldots, t_m, n \) is jointly sufficient for \( p_1, \ldots, p_m \).

The prior of \( p_1, \ldots, p_m \) should be

\[
g'(p_1, \ldots, p_m) \propto p_1^{n_1-1}(1-p_1)^{n_1-n_1-1} \cdots p_m^{n_m-1}(1-p_m)^{n_m-n_m-1}
\]

where \( m \) is the number of groups which we will assume it known and categorized by some characteristic such as age at marriage or parity if \( p_i \) is replaced in \( p_x \).

Then the posterior would be

\[
g''(p_1, p_2, \ldots, p_m | t_1, \ldots, t_m)
\]

\[
\propto p_1^{n_1+n_2-1}(1-p_1)^{n_1-n_1-1} \cdots p_m^{n_m+n_2-1}(1-p_m)^{n_m-n_m-1}
\]

where \( n_1 + n_2 + \cdots + n_m = n \), all \( n_i \)'s are known.

This result can be easily derived by the independent of \( t_i \)'s. Hence the unconditional distribution of \( t_1, \ldots, t_m \) can be expressed by

\[
g(t_1, \ldots, t_m | \nu_{01}, \nu_{02}, \ldots, \nu_{0m}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m})
\]

\[
= \prod_{j=1}^{m} \frac{(v_{1j} - 1)! (v_{0j} - n_j - 1)! (n_j ! (n_j - n_{0j} - 1) ! (n_{0j} ! (v_{0j} - 1) ! (v_{1j} - n_{0j} - 1) ! (n_{0j} ! (v_{0j} - 1) ! (v_{1j} - n_{0j} - 1) !)
\]

Bibliography