Mappings Generating Upper-semicontinuous Decompositions of Spaces with Coherent Topologies

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1. Introduction

In this paper we generalize the concepts of compactly generated spaces (or Hausdorff k-spaces) and reflexive compact mappings. Using these concepts we obtain sufficient conditions for mappings to generate upper-semicontinuous (u.s.c.) decompositions of certain types of spaces with coherent topologies. We also show that some of the results generalize similar situations which are given previously.

2. Terminologies

A k-space $X$ is a topological space having a topology coherent with the collection of its closed compact subsets; i.e., if a subset $A$ of $X$ intersects each closed compact set in a closed set, then $A$ is closed [4]. A Hausdorff k-space is said to be compactly generated [6]. A topological space $X$ is said to be locally paracompact if every point of $X$ has a closed neighborhood which is a paracompact subspace of $X$ [2]. Throughout this paper $X$ and $Y$ will represent topological spaces and $f$ will be a mapping (continuous function) of $X$ into $Y$. A mapping $f$ of $X$ into $Y$ is said to be reflexive compact provided that $f^{-1}(C)$ is compact for every compact subset $C$ of $X$ [3]. All terminologies given above will be generalized in Sections 3 and 4. A mapping $f$ of $X$ into $Y$ is said to generate an u.s.c. decomposition of $X$ if for every open set $U$ of $X$, the union $V$ of point inverses $f^{-1}(y)$ contained in $U$ is an open subset of $X$.

3. $P$-generated spaces

Throughout this paper, a topological property $P$ is said to be admissible if it is inherited by closed sets.

3.1. Definition. Let $X$ be a space and $P$ an admissible property. A $P$-set in $X$ is a closed subset of $X$ which possesses the property $P$. $X$ is said to be $P$-generated if it has a topology coherent with the collection of its $P$-sets; i.e., if a subset $A$ of $X$ intersects each $P$-set in a closed set, then $A$ is closed.

3.2. Definition. Let $P$ and $Q$ be admissible properties. We shall write $P \subset Q$ if and only if every $P$-set is a $Q$-set.

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For instance, let $P$ and $Q$ be the compactness and the paracompactness, respectively. Then $P \subset Q$ in a Hausdorff space.

3.3. **Definition.** A neighborhood which is a $P$-set will be called *$P$-neighborhood*. A space $X$ is said to be *locally $P$* provided that every point of $X$ has a $P$-neighborhood.

It is obvious that if $P \subset Q$ and $X$ is locally $P$, then $X$ is locally $Q$.

3.4. **Proposition.** A locally $P$-space is $P$-generated.

**Proof.** Let $B$ be a non-closed subset of $X$ and suppose $x$ is an accumulation point of $B$ which does not belong to $B$. Since $X$ is locally $P$ there is a $P$-neighborhood $U$ of $x$ and the intersection $B \cap U$ is not closed because $x$ is an accumulation point but not a member of $B \cap V$.

3.5. **Corollary.** A locally compact Hausdorff space is compactly generated [4].

3.6. **Proposition.** If a space $X$ is $P$-generated and $P \subset Q$, then $X$ is $Q$-generated.

**Proof.** Let $B$ be a non-closed subset of $X$. If $X$ is $P$-generated there is a $P$-set $V$ of $X$ such that $V \cap B$ is not closed. Since $P \subset Q$, $V$ is also $Q$-set.

We can define an analogue to compactly generated spaces.

3.7. **Definition.** A *paracompactly generated space* is a Hausdorff space having a topology coherent with the collection of its closed paracompact subsets.

From Propositions 3.4 and 3.6 we obtain

3.8. **Corollary.** A Hausdorff space which is either locally paracompact or compactly generated is paracompactly generated.

It is established that a Hausdorff space which satisfies one of the following conditions is compactly generated.

(a) locally compact [4]

(b) the first axiom of countability [4]

(c) product of a compactly generated space and a locally compact Hausdorff space [1, 6]

Hence, from the latter part of Corollary 3.8, we obtain

3.9. **Corollary.** A Hausdorff space which satisfies one of the conditions (a), (b) and (c) is paracompactly generated.

In summary we have

$$
\text{compact} \implies \text{locally compact} \implies \text{compactly generated} \\
\implies \text{paracompact} \implies \text{locally paracompact} \implies \text{paracompactly generated} \\
\implies \text{normal} \implies \text{completely regular} \implies \text{Hausdorff}
$$
in the class of Hausdorff spaces and we know none of the arrows can be reversed.

Especially, the following example in [1] which shows the product of two compactly generated spaces need not be a compactly generated space serves as a paracompactly generated space which is not compactly generated.

3.10. Example. Let \( X \) be the dual space of an infinite dimensional Fréchet space with the compact-open topology is compactly generated space which is not locally compact. \( F = C(X, [0, 1]) \) with the compact-open topology is metrizable. However \( X \times F \) is not compactly generated. It follows from [5, Prop.4] that \( X \times F \) is paracompact which implies it is paracompactly generated.

4. Reflexive \( P \)-mappings

4.1. Definition. Let \( f \) be a mapping of \( X \) into \( Y \). \( f \) is called a reflexive \( P \)-mapping provided that \( f^{-1}(P) \) is a \( P \)-set for every \( P \)-set \( P \) of \( X \).

With the above definition we obtain a sufficient condition for mappings to generate u.s.c. decompositions of \( P \)-generated spaces.

4.2. Theorem. Let \( X \) be a \( P \)-generated space and \( f \) a mapping of \( X \) into \( Y \). If \( f \) is a reflexive \( P \)-mapping then \( f \) generates an upper-semicontinuous decomposition of \( X \).

This is a generalization of [3, Theorem 1] and the following proof is a slight modification of that in [3].

Proof. Let \( U \) be an open set in \( X \) containing a point inverse and \( V \) the union of the point inverses contained in \( U \). We show that \( U - V \) is closed in \( X \) which in turn implies \( V \) is open in \( X \) and consequently \( f \) generates an u.s.c. decomposition. Let \( P \) be a \( P \)-set in \( X \) such that \( H = (U - V) \cap P \neq \emptyset \). For each point \( x \) in \( H \), \( f^{-1}(x) \cap (X - U) \neq \emptyset \). Thus we obtain

\[
f^{-1}(P) \cap (X - U) = f^{-1}(P \cap U) \cap (X - U).
\]

The right member is a \( P \)-set so is the left member. Denoting the set in (1) above by \( M \) we obtain

\[
f^{-1}(M) \cap U = f^{-1}(P \cap U) \cap (U - V).
\]

The left member is a \( P \)-set so is the right member. The set \( P \cap (U - V) \) is closed for it is the intersection of the \( P \)-sets \( P \) and \( f^{-1}(P \cap U) \cap (U - V) \). Thus \( X \) being \( P \)-generated implies \( U - V \) is a closed subset of \( X \).

Since the closedness is admissible and every closed mapping is reflexive closed, we obtain

4.3. Corollary. Let \( f \) be a mapping of \( X \) into \( Y \). If \( f \) is closed then \( f \) generates an upper-semicontinuous decomposition of \( X \).
It is well-known the converse of Corollary 4.3 also holds.

4.4. Corollary. Let $X$ be a compactly generated space and $f$ a mapping of $X$ into $Y$. If $f$ is reflexive compact, then $f$ generates an upper-semicontinuous decomposition of $X$ [3. Theorem 1].

4.5. Corollary. Let $X$ be a paracompactly generated space and $f$ a mapping of $X$ into $Y$. If $f$ is reflexive paracompact, then $f$ generates an upper-semicontinuous decomposition of $X$.

Combining Corollary 3.8 with Corollary 4.5 we obtain the following corollaries.

4.6. Corollary. Let $X$ be a compactly generated space and $f$ a mapping of $X$ into $Y$. If $f$ is reflexive paracompact, then $f$ generates an upper-semicontinuous decomposition of $X$.

4.7. Corollary. Let $X$ be a locally paracompact Hausdorff space and $f$ a mapping of $X$ into $Y$. If $f$ is reflexive paracompact, then $f$ generates an upper-semicontinuous decomposition of $X$.

References


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