On Semi-simplicity and Weak Semi-simplicity

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It is well known that the radical $J(A)$ of a ring $A$ is the intersection of the modular maximal right ideals of $A$ and a ring $A$ is semi-simple if and only if $J(A) = (0)$. The purpose of this note is to prove that a semi-simple ring is weakly semi-simple and that if $A$ is a commutative ring, then $(0)$ is almost maximal if and only if $A$ is an integral domain. The notation and terminology of this note are based on [1].

**Definition 1.** If $I$ is a proper right ideal of a ring $A$ then $I$ is almost maximal provided that

1. if $J_1$ and $J_2$ are right ideals of $A$ and $J_1 \cap J_2 = I$, then $J_1 = I$ or $J_2 = I$, i.e., $I$ is meet-irreducible,
2. if $a \in A$ and $[I:a] \supset I$, then $a \in I$,
3. if $J$ is a right ideal of $A$, $J \supset I$, then $N(I) \cap J \supset I$, where $N(I) = \{a \in A : aI \subseteq I\}$, and if $a \in A$ such that $[J:a] \supset I$ then $[J:a] \supset I$.

**Theorem 1.** If $A$ is a commutative ring, then $(0)$ is an almost maximal ideal if and only if $A$ is an integral domain.

**Proof.** Let $b$ be a nonzero element of $A$ and $ab = 0$ for some $a \in A$. Then $[(0):a] \supset b$, i.e., $[(0):a] \supset (0)$. Since $(0)$ is almost maximal, $a \in (0)$, i.e., $a = 0$. Hence $A$ is an integral domain. Conversely, if $J_1$ and $J_2$ are right ideals of $A$, $J_1 \supset (0)$ and $J_2 \supset (0)$, then there exist nonzero elements $a$ in $J_1$ and $b$ in $J_2$ such that $ab \neq 0$, since $A$ is an integral domain, and $ab \in J_1 \cap J_2$, i.e., $J_1 \cap J_2 \supset (0)$. Now suppose $a \in A$ and $[(0):a] \supset (0)$, then there exists a nonzero element $b$ in $A$ such that $ab = 0$. Since $A$ is an integral domain, $a = 0$, i.e., $a \in (0)$. Again assume $J \supset (0)$. Since $N((0)) = A$, $J \cap N((0)) = J \supset (0)$. And if $a \in A$, then $[J:a] \supset J \supset (0)$. Therefore $(0)$ is almost maximal.

The proofs of Theorem 2 and Corollary depend upon the following lemmas proved in [1].

**Lemma 1.** Let $W(A)$ be the weak radical of a ring $A$. If $W(A) \neq A$, then $W(A)$ is the intersection of almost maximal right ideals of $A$.

**Lemma 2.** If $A$ is a ring and $B$ is a two-sided ideal of $A$, then $W(B) = W(A) \cap B$.

**Definition 2.** A ring is called weakly semi-simple if and only if its weak radical is zero ideal.

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THEOREM 2. If a ring $A$ is semi-simple, then $A$ is weakly semi-simple.

Proof. We show that a modular maximal right ideal $I$ of $A$ is almost maximal, from which it follows that $W(A) \subseteq J(A)$ by Lemma 1, and the theorem will be proved. Assume $J$ is a right ideal of $A$ and $J \supseteq I$. Then $J = A$ since $I$ is maximal. Therefore $I$ is meet-irreducible in $A$. Now suppose $a \in A$ and $[I:a] \supseteq I$. Then $[I:a] = A$, i.e., $aA \subseteq I$. If $a \in I$, then $a + I$ is a generator of a strictly cyclic $A$-module $A - I$. For any $b$ in $A$, there exists an element $c$ in $A$ such that $(a + I)c = b + I$. Then $b \in I$, since $ac - b \in I$ and $ac \in I$, which is impossible. Hence $a \not\in I$. Let $e$ be a left identity modulo $I$. Then $I \subseteq N(I)$, $e \in N(I)$ and $e \in I$, i.e., $N(I) \supseteq I$. If $J \supseteq I$, then $J \cap N(I) = N(I) \supseteq I$. Assume $J \supseteq I$ and $e \in A$ such that $[J:a] \supseteq I$. Since $[J:a] = [A:a]$, $[J:a] \ni e$ and thus $[J:a] \supseteq I$. Thus the theorem is proved.

COROLLARY. Let $A$ be a ring and $B$ be an ideal of $A$ such that $A$ and $B$ have the same radical. Then $A$ and $B$ have the same weak radical.

Proof. Since $J(A) = J(B) = J(A) \cap B$, $J(A) \subseteq B$ and thus $W(A) \subseteq B$. Therefore, by Lemma 2, $W(B) = W(A) \cap B = W(A)$.

Reference


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