NOTE ON THE CHARACTERIZATIONS OF MINIMAL $T_0$ AND $T_D$ SPACES

By Ki-Hyun Pahk

Introduction

Given a set $X$ and the lattice of all topologies on the set. We will investigate properties of the minimal $T_0$ topology and minimal $T_D$ topology on this set. In the paper [4], M.P. Berri discussed minimal properties of Hausdorff spaces, Frechet spaces, completely regular spaces, normal spaces, and locally compact spaces.

In general, the terminology of this paper will coincide with the terminology found in [4]. In the comparison of topologies, a topology $\mathcal{T}$ will be weaker than a topology $\mathcal{T}'$ if $\mathcal{T}$ is a subfamily of $\mathcal{T}'$. In this paper, the notation $\mathcal{Y}$ denotes the family of all closed sets on $X$.

DEFINITION 1. A topology $\mathcal{F}$ on a set $X$ is said to be weaker than a topology $\mathcal{F}'$ on $X$, if, for each closed set $F$ in $(X, \mathcal{F})$, $F$ is also a closed set in $(X, \mathcal{F}')$.

§ 1. Characterizations of minimal $T_0$ space.

1.1. DEFINITION A topological space $(X, \mathcal{F})$ is said to be minimal $T_0$ space, if $\mathcal{F}$ is $T_0$ topology and there exists no $T_0$ topology on $X$ strictly weaker than $\mathcal{F}$.

1.2. LEMMA Let $(X, \mathcal{F})$ be $T_0$ space on an infinite set $X$. If there exist $A$ and $B$ in $\mathcal{F}$ such that $a=A$, $b=B$ and $A\cap B=\emptyset$, then $(X, \mathcal{F})$ is not minimal $T_0$ space.

PROOF Suppose $\mathcal{F}$ is minimal $T_0$ topology, then there exists $\mathcal{F}'$ which is proper subfamily of $\mathcal{F}$ as the following:

$$\mathcal{F}' = \{T | T = A \cup A_\alpha (b \in A_\alpha) \text{ or } T = A_\beta (b \in A_\beta), A_\alpha, A_\beta \in \mathcal{F}\}.$$ 

Now, we prove for $\mathcal{F}'$ to be a topology as the following:

$$\left(\bigcup_{\alpha=1}^{n} (A \cup A_\alpha)\right) \cup \left(\bigcup_{\beta=1}^{m} A_\beta\right) = \left(A \cup \left(\bigcup_{\alpha=1}^{n} A_\alpha\right)\right) \cup \left(\bigcup_{\beta=1}^{m} A_\beta\right)$$

$$= A \cup \left(\bigcup_{\alpha=1}^{n} A_\alpha\right) \cup \left(\bigcup_{\beta=1}^{m} A_\beta\right).$$
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where

\[
\bigcup_{\alpha=1}^{n} A_\alpha \cup \left( \bigcup_{\beta=1}^{m} A_\beta \right) \in \mathcal{F}, \quad \mathcal{B} \left( \bigcup_{\alpha=1}^{n} A_\alpha \right) \cup \left( \bigcup_{\beta=1}^{m} A_\beta \right)
\]

Hence, the finite union of closed sets in \( \mathcal{F} \) is contained also in \( \mathcal{F}' \). And \(( \bigcap_{\alpha \in \mathcal{A}} (A_\alpha \cup A_{\alpha'}) \cap (\bigcap_{\beta \in \mathcal{B}} A_\beta) \) doesn’t contain \( b \), so that itself is contained in \( \mathcal{F}' \).

Here \( \mathcal{F}' \) contains \( \phi \) and \( X \) clearly.

Next, we prove for \( \mathcal{F}' \) to be \( T_0 \) topology as the following.

We consider the three cases i.e. (1) the case of \( \overline{x} \neq \overline{y} \), (2) the case of \( \overline{x} = \overline{y} \), \( \overline{x} \neq \overline{y} \), and (3) the case of \( \overline{x} \neq \overline{y} \), \( \overline{x} = \overline{y} \), for arbitrary elements \( x, y \) in \( (X, \mathcal{F}) \).

Here, \( \overline{x} \) denotes the closure of \( x \) in \( \mathcal{F} \), and \( \overline{x}' \) denotes the closure of \( x \) in \( \mathcal{F}' \).

1.3. LEMMA Let \( (X, \mathcal{F}) \) be a \( T_0 \) space on infinite set \( X \). If there exist \( A \) and \( B \) in \( (X, \mathcal{F}) \) such that \( a = A \), \( b = B \) and \( A \cap B = C \), where \( C \ni \phi \), \( C \ni A \) and \( C \ni B \), then \( (X, \mathcal{F}) \) is not minimal \( T_0 \) space.

PROOF Suppose \( \mathcal{F} \) is minimal \( T_0 \) topology, then there exists \( \mathcal{F}' \) which is proper subfamily of \( \mathcal{F} \) as the following:

\[
\mathcal{F}' = \{ T | T = A_\alpha \cup A_\beta (\mathcal{B} \in \mathcal{F}) \}.
\]

Therefore, as well as the method in proof of 1.2 Lemma, we can see easily that \( \mathcal{F}' \) is a \( T_0 \) topology.

1.4. THEOREM Let \( \mathcal{F} \) be a minimal \( T_0 \) topology on an infinite set \( X \). \( \mathcal{F} = \{ a_\alpha : \alpha \in \mathcal{A} \} \), then arbitrary two elements \( A_\alpha, A_\beta \) in \( \mathcal{F} \) have the following relations

\[ A_\alpha \supset A_\beta \quad \text{or} \quad A_\alpha \subseteq A_\beta. \]

PROOF. Suppose that above result does not happen from the assumption, then it arises following two cases, but all the cases come to have contradiction.

Now, we consider the case of \( A_\alpha \cap A_\beta = \phi \). There exist \( a, b : \alpha \in A_\alpha, \beta \notin A_\beta \), and we have \( \overline{a} \cap \overline{b} = \phi \). Hence \( (X, \mathcal{F}) \) is not minimal \( T_0 \) space by 1.2 Lemma.

Next, we consider the case of \( A_\alpha \cap A_\beta = A \), where \( A \) is different from \( \phi \), \( A_\alpha \) and \( A_\beta \). There exist \( a \) and \( b \):

\[
a \in A_\alpha - A \quad \text{and} \quad b \in A_\beta - A.
\]
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Hence $(X, \mathcal{T})$ is not minimal $T_0$ space by 1.3. Lemma or 1.2. Lemma respectively.

1.5. THEOREM  A necessary and sufficient condition that a $T_0$ space $(X, \mathcal{T})$ be minimal is that $\mathcal{T}$ satisfies the following properties (i), (ii).

(i) Every two closed sets in $\mathcal{T} = \{A_\alpha : \alpha \in \Delta\}$ have the relation $A_\alpha \subseteq A_\beta$ or $A_\alpha \supseteq A_\beta$.

(ii) If we put $A_\alpha \not\subseteq X$ and $A_\alpha \not\subseteq \emptyset$, then $A_\alpha$ is represented by $A_\alpha = a_\alpha$ or $A_\alpha = \bigcap_{\beta \in \Delta} \bar{a}_\beta$.

PROOF  At first we will show that the conditions (i), (ii) are necessary. We put $\mathcal{T}' = \mathcal{T} - \{A_\alpha : \alpha \in \Delta', \Delta' \subseteq \Delta\}$,
and we show as following, that $(X, \mathcal{T}')$ is not $T_0$ space:

Now we consider the case of $A_\alpha = a_\alpha (\alpha \in \Delta')$ in $\mathcal{T}$.
Let $a'_\alpha = A_\beta$ in $\mathcal{T}'$, then $A_\alpha \subseteq A_\beta$. If $A_\beta = a_\beta$ in $\mathcal{T}'$, then $A_\beta = \bar{a}_\beta$. Hence $a'_\alpha$ is equal to $a_\beta$ in $\mathcal{T}'$, which is contrary. And if $A_\beta$ is not a point closure in $\mathcal{T}'$, then $A_\alpha \not\subseteq A_\beta$ and $A_\alpha \subseteq A_\beta$. Hence there is an element $x$ such that $x \in A_\beta - A_\alpha$.
Since $a_\alpha = A_\beta$, $A_\beta$ is the least closed set containing $a_\alpha$ in $\mathcal{T}'$. Since $x \in A_\beta$, let $x' = A_\alpha$, then $A_\alpha = A_\beta$. On the other hand, since $x \not\subseteq A_\beta$, we have $a_\alpha \not\subseteq A_\beta$, $a'_\alpha = A_\beta$ and $A_\beta \subseteq A_\beta$. Hence we have $x = A_\beta$ and $x' = a'_\alpha$, which is contrary.

And, we consider the case of $A_\alpha = \bigcap_{\beta \in \Delta} a_\beta$ in $\mathcal{T}$.
Here, we can see $A_\alpha \in \mathcal{T}$ and $A_\alpha \in \mathcal{T}'$. On the other hand, since $a_\beta \in \mathcal{T}'$, we have $\bigcap_{\beta \in \Delta} a_\beta = A_\alpha \in \mathcal{T}'$.

Therefore this case can't arise.

Next, we show that the conditions (i), (ii) are sufficient.
Let $\mathcal{T}$ be a minimal $T_0$ topology which does not satisfies (i) or (ii). But by 1.4. theorem, we know that $\mathcal{T}$ satisfies (i). Hence, this assumption represents that there exists an element $A_\beta$ of $\mathcal{T}$ which doesn't satisfy (ii) only. We put $\mathcal{T}' = \mathcal{T} - A_\beta$. 
then clearly $\mathcal{F}'$ is a topology and $(X, \mathcal{F}')$ is $T_0$ space.

It is contrary.

1.6. REMARK As following example, we know that $(X, \mathcal{F})$ is not minimal $T_0$ space, if $\mathcal{F}$ is a family of non-linearly ordered sets. Now, let $X$ be set $\{a, b, c\}$ and $\mathcal{F}$ be a closed family in $X$: $\phi$, $\{c\}$, $\{b, c\}$, $\{a, c\}$, $X$, then $\mathcal{F}$ is a topology clearly. And point closures of each elements are different mutually, since $a = \{a, c\}$, $b = \{b, c\}$, $c = \{c\}$. Hence $(X, \mathcal{F})$ is $T_0$ space, but $\mathcal{F}$ is a family of non-linearly ordered sets. Here we put that $\mathcal{F}'$ is a family of closed sets excepting $\{a, c\}$ in $\mathcal{F}$, then $\mathcal{F}'$ is also $T_0$ topology and $\mathcal{F}'$ is strictly weaker than $\mathcal{F}$.

Finally $(X, \mathcal{F})$ is not minimal $T_0$ space.

1.7. REMARK On 1.5. theorem, the condition (i) only is not sufficient for $T_0$ space to be minimal. As an example, we put

$X=\{[1, 2], 3, 4, 5, 6\ldots\}$,
and $\mathcal{F}$ is a family of closed sets in $X$ such that

$X$, $\{[1, 2], 3, 4, 5, \ldots\}$, $\{[a_i, 2], 3, 4, 5, \ldots\}$,
$\{2, 3, 4, 5, \ldots\}$, $\{3, 4, 5, \ldots\}$, $\ldots$,

where $a_i$ is a real number such that $1 < a_i < 2$.

Then $\mathcal{F}$ is a family of linearly ordered sets. Now $(X, \mathcal{F})$ is $T_0$ space clearly, but we put that $\mathcal{F}'$ is a family of closed sets excepting $\{[1, 2], 3, 4, 5, \ldots\}$ in $\mathcal{F}$, then $\mathcal{F}'$ is also $T_0$ topology and $\mathcal{F}'$ is strictly weaker than $\mathcal{F}$. Finally, $(X, \mathcal{F})$ is not minimal $T_0$ space.

§ 2. Characterizations of minimal $T_D$ space.

2.1. DEFINITION A topological space $(X, \mathcal{F})$ is said to be minimal $T_D$-space, if $\mathcal{F}$ is $T_D$ topology and there exists no $T_D$ topology on $X$ strictly weaker than $\mathcal{F}$.

As the ground of this part, we begin with the following theorem in paper [1].

2.2. THEOREM Let $\mathcal{F}$ be a topology in a set $X$.

(1) If $(X, \mathcal{F})$ is a $T_1$ space, then $(X, \mathcal{F})$ is $T_D$ space.

(2) If $(X, \mathcal{F})$ is a $T_D$ space, then $(X, \mathcal{F})$ is $T_0$ space.
2.3. THEOREM Let $\mathcal{T}$ be a minimal $T_D$ topology on an infinite set $X$: $\mathcal{T} = \{ A_\alpha : \alpha \in \Delta \}$, then arbitrary elements $A_\alpha$, $A_\beta$ in $\mathcal{T}$ have following relations

$$A_\alpha \supset A_\beta \text{ or } A_\alpha \subset A_\beta.$$ 

We can easily see above theorem by following 2.4. Lemma and 2.5. Lemma.

2.4. LEMMA Let $(X, \mathcal{T})$ be $T_D$ space on an infinite set $X$. If there exist $A$ and $B$ in $(X, \mathcal{T})$ such that $a = A$, $b = B$ and $A \cap B = \emptyset$, then $(X, \mathcal{T})$ is not minimal $T_D$ space.

2.5. LEMMA Let $(X, \mathcal{T})$ be a $T_D$ space on infinite set $X$. If there exist $A$ and $B$ in $(X, \mathcal{T})$ such that $a = A$, $b = B$ and $A \subseteq C = \emptyset$, where $C \subseteq A$ and $C \not\subseteq B$, then $(X, \mathcal{T})$ is not minimal $T_D$ space.

We can easily see above Lemma 2.4 and 2.5 in the similar method with the proof of 1.3 and 1.4.

2.6. THEOREM A necessary and sufficient condition that a $T_D$ space $(X, \mathcal{T})$ be minimal is that $\mathcal{T}$ satisfies following properties (i), (ii).

(i) Every two closed sets in $\mathcal{T} = \{ A_\alpha : \alpha \in \Delta \}$ has relation

$$A_\alpha \supset A_\beta \text{ or } A_\alpha \subset A_\beta.$$ 

(ii) If we put $A_\alpha = X$ and $A_\alpha = \emptyset$, then $A_\alpha$ satisfies

$$A_\alpha = a_\alpha, \quad A_\alpha = a_\alpha - a_\alpha \text{ or } A_\alpha = \bigcap_{\alpha \in \Delta} a_\alpha \cap \bigcap_{\beta \in \Delta} (a_\beta - a_\beta).$$

PROOF At first we will show that the conditions (i), (ii) are necessary. We put

$$\mathcal{T}' = \mathcal{T} - \{ A_\beta : \beta \in \Delta, \ A_\beta \subseteq \Delta \},$$

and we show as following, that $(X, \mathcal{T}')$ is not $T_D$ space;

Now we consider the case of $A_\beta = a_\beta$ ($\beta \in \Delta$) in $\mathcal{T}$. Let $a_\beta' = A_\lambda$ in $\mathcal{T}'$ then $A_\beta \subseteq A_\lambda$. If $A_\beta = a_\lambda$ in $\mathcal{T}'$ then $A_\beta = a_\alpha'$ in $\mathcal{T}'$.

Hence $a_\beta'$ is equal to $a_\alpha'$ in $\mathcal{T}'$, which is contrary. Therefore, we can see that $(X, \mathcal{T}')$ is not $T_0$ space, i.e., $(X, \mathcal{T}')$ is not $T_D$ space. And if $A_\lambda = a_\alpha - a_\lambda$ in $\mathcal{T}$, then $A_\lambda \not\subseteq A_\beta$ and $A_\lambda \supset A_\beta$. Hence there exists an element $x$ such that $x \not\in A_\lambda - A_\beta$ and $x \not\subseteq a_\beta$. Since we put $x = A_\mu$, then $A_\mu \subseteq A_\lambda$. On the otherhand, $a_\beta$
is an element of \( A_{\mu} \). Hence we have \( a'_\beta \subseteq A_{\mu} \), i.e., \( A_\mu \subseteq A_{\mu} \). Therefore we may have \( A_\lambda = A_{\mu} \), i.e. \( a'_\beta = x' \). We can see that \((X, \mathcal{T}')\) is not \( T_0 \) space, i.e., \( (X, \mathcal{T}') \) is not \( T_D \) space.

And we consider the case \( A_\beta = a_\beta - a_\beta(\beta \epsilon \Delta') \) in \( \mathcal{T} \). From the first case, we can see that all \( a_\beta \) contain \( \mathcal{T}' \). Since \( \mathcal{T}' \) is \( T_D \) topology, all \( a_\beta - a_\beta \) are the elements of \( \mathcal{T}' \). Hence this case does not arise.

Next we consider the case \( A_\beta = \bigcap_{\alpha \in d} \bigcap_{\tau \in d} (a_\tau - a_\tau) (\beta \epsilon \Delta') \) in \( \mathcal{T} \). Here \( \bigcap_{\alpha \in d} a_\alpha \) does not fall away in \( \mathcal{T}' \) under the first case, and similarly for \( \bigcap_{\tau \in d} (a_\tau - a_\tau) \) under the second case. Therefore, \( A_\beta \) must be an element of \( \mathcal{T}' \). It is contrary.

The proof of the sufficiency is the similar method in proof of 1.5. Theorem.

Department of Mathematics
Kyungpook University

REFERENCES