ON PRECORRECT UNIFORM SPACES

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A subset \( \mathcal{U} \) of the power set of \((X \times X)\) is a \textit{precorrect uniformity} on \(X\) iff \( \mathcal{U} \) satisfies (\(A_1\)) \( U \in \mathcal{U} \) iff \( U = U^{-1}, U \supseteq \Delta \), and \( U \) contains a member of \( \mathcal{U} : (A_2) \)

For every \( A \subseteq X \) and \( U, V \in \mathcal{U} \) there exists a \( W \in \mathcal{U} \) such that \( W \cdot A \subseteq U \cdot A \cap V \cdot A : (A_3) \)

(\(A_2\)) For every \( A \subseteq X \) and \( U \in \mathcal{U} \) there exists \( V, W \) in \( \mathcal{U} \) such that \((W \cup V) \cdot A \subseteq U \cdot A \).

A relation \( \delta \) on \( P(X) \), the power set of \(X\), is a \textit{proximity} on \(X\) iff \( \delta \) satisfies:

(\(P_1\)) \( A \delta B \implies B \delta A \); (\(P_2\)) \( C \delta (A \cup B) \) iff either \( \delta CA \) or \( C \delta B \); (\(P_3\)) \( \phi \delta A \) for every \( A \subseteq X \); (\(P_4\)) \( x \delta x \) for all \( x \in X \); (\(P_5\)) \( A \delta B \) implies the existence of \( C \) and \( D \) such that \( C \cap D = \phi \), and \( A \delta (X - D) \), \( B \delta (X - D) \)

**THEOREM 1.** Let \( \mathcal{U} \) be a subset of the power set of \((X \times X)\). Suppose for each \( U \in \mathcal{U} \) \( U = U^{-1} \). Define a relation \( \delta(\mathcal{U}) \) on \( P(X) \) by \( A \delta(\mathcal{U}) B \) iff \( U \cdot A \cap B = \phi \) for all \( U \in \mathcal{U} \). Then \( \delta(\mathcal{U}) \) satisfies (\(P_1\)), (\(P_2\)), (\(P_3\)), (\(P_4\)) and (\(P_5\)) iff \( \mathcal{U} \) satisfies (\(A_1^*\)): \( U \in \mathcal{U} \) implies \( U \supseteq \Delta, (A_2) \) and (\(A_3\)).

A proof of Theorem 1 is given in [1].

If we are given \( \delta \), a proximity on \(X\), then the class of precorrect uniformities \( \mathcal{U} \) on \(X\) such that \( \delta(\mathcal{U}) = \delta \) is called a \textit{proximity class of precorrect uniformities} on \(X\) and is denoted by \( II(\delta) \).

**THEOREM 2.** Let \((X, \delta)\) be a proximity space. Then \( II(\delta) \) contains one and only one totally bounded symmetric uniformity.

**PROOF.** This is an immediate consequence of Theorem 21.20 in [6].

**THEOREM 3.** Let \((X, \delta)\) be a proximity space. Then \( II(\delta) \) contains a maximum and a minimum.

**PROOF.** For all \( A, B \) in \( P(X) \) let \( U_{A, B} = (X \times X) - ((A \times B) \cup (B \times A)) \). It is easy to show by Theorem 1 that \( \mathcal{B} = \{ U_{A, B} | A \delta B \} \) is a base for a precorrect uniformity \( \mathcal{U}_1(\delta) \) on \(X\) such that \( \mathcal{U}_1(\delta) \) is the least element in \( II(\delta) \). Also, it is easily shown by Theorem 1 that the union of an arbitrary family of members of \( II(\delta) \) is a base for a precorrect uniformity on \(X\) that is a member of \( II(\delta) \); consequently \( II(\delta) \) has a maximum element.
THEOREM 4. If \( \delta \) is the usual proximity for the reals, \( X \), then \( II(\delta) \) contains at least two distinct precompact precorrect uniformities that have an open base.

PROOF. Let \( \gamma = \{ U_{A,B} | A \bar{B} \} \). Let \( \mathcal{B} = \{ \text{all finite intersections of members of } \gamma \} \).

It can be shown by Theorem 1 that \( \mathcal{B} \) is a base for a precompact symmetric uniformity \( \mathcal{U}_2(\delta) \) on \( X \) such that if \( \mathcal{U}_1(\delta) \) is the uniformity that was constructed in Theorem 3 then \( \mathcal{U}_1(\delta) \) is properly contained in \( \mathcal{U}_2(\delta) \). It is easily shown that both \( \mathcal{U}_1(\delta) \) and \( \mathcal{U}_2(\delta) \) are totally bounded and have an open base (cf. [5]).

We say that a filter in the precorrect uniform space \( (X, \mathcal{U}) \) is weakly Cauchy iff for every \( U \in \mathcal{U} \) there exists an \( x \in X \) such that \( U[x] \in \mathcal{F} \). Also, \( (X, \mathcal{U}) \) is complete iff every weakly Cauchy filter on \( (X, \mathcal{U}) \) has a cluster point in \( X \).

THEOREM 5. If \( (X, \mathcal{T}) \) is a connected completely regular topological space, then there exists a precompact precorrect uniformity \( \mathcal{U} \) on \( X \) with an open base such that \( \mathcal{T}(\mathcal{U}) = \mathcal{T} \) and every filter is weakly Cauchy.

PROOF. Let \( \bar{\delta} \) be a proximity on \( X \) such that \( \mathcal{T}(\bar{\delta}) = \mathcal{T} \). Let \( \mathcal{U}_1(\bar{\delta}) \) be an element of \( II(\bar{\delta}) \). Note that \( \mathcal{U}_1(\bar{\delta}) \) exists by Theorem 3. Let \( U \in \mathcal{U}_1(\bar{\delta}) \).

Then there exist sets \( A \subset X \) and \( B \subset X \) such that \( U \supset U_{A,B} \supset U_{\overline{A},B} \). But since \( \mathcal{T} \) is connected, there exists \( x_0 \in (X - (A \cup B)) \) so that \( U_{\overline{A},B}[x_0] = X \). Hence every filter on \( X \) is weakly Cauchy with respect to \( \mathcal{U} \).

THEOREM 6. A precorrect space is compact iff it is complete and precompact.

PROOF. This is an immediate consequence of Theorem 4 in [4] and the easily established fact namely that every precorrect uniform space is a symmetric generalized uniform space (cf. [1]).

THEOREM 7. A completely regular topological space is compact iff it is complete with respect to every compatible precorrect uniformity on \( X \).

PROOF. This is an immediate consequence of the Lemma on page 5 in [3] and Theorem 6 above.
REFERENCES


