NOTE ON INFINITESIMAL $\eta$-CONFORMAL AND CL-
TRANSFORMATIONS OF SPECIAL CONTACT
METRIC SPACES

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Y. Tashiro and S. Tachibana showed some characteristic properties of Fubinian and $C$-Fubinian manifolds in their paper [1], where the notion of $C$-loxodromes was introduced in an almost contact manifold with affine connection. Recently H. Mizusawa defined an infinitesimal $\eta$-conformal transformation in a contact metric space [2]. K. Takamatsu and H. Mizusawa have shown some relations in a compact normal contact metric space under an infinitesimal $CL$-transformation [3].

In the previous paper [4], we have obtained that an infinitesimal $CL$-transformation in a normal contact and $K$-contact metric space had some analogous properties of [3]. In this paper, we study on infinitesimal $\eta$-conformal and $CL$-transformations in $K$-contact and normal contact metric spaces.

§ 1. Preliminaries

An $n$ (=2$m$+1)-dimensional differentiable manifold $M$ of class $C^\infty$ with ($\phi$, $\xi$, $\eta$, $g$)-structure (or an almost contact metric structure) has been defined by S. Sasaki [5]. By definition it is a manifold with tensor fields $\phi^i_j$, $\xi^i_j$, $\eta_i$ and so called an associated Riemannian metric tensor $g_{ji}$ defined over $M$ which satisfy the following relations:

\begin{align*}
(1.1) & \quad \xi^i_j \eta_i = 1, \\
(1.2) & \quad \operatorname{rank} |\phi^i_j| = n - 1, \\
(1.3) & \quad \phi^i_j \xi^j = 0, \\
(1.4) & \quad \phi^i_j \eta_i = 0, \\
(1.5) & \quad \phi^i_j \phi^j_r = -\delta^i_j + \xi^i_j \eta_j, \\
(1.6) & \quad g_{ji} \xi^j = \eta_i, \\
(1.7) & \quad g_{ji} \phi^i_h \phi^j_k = g_{kh} - \eta_k \eta_h.
\end{align*}

On the other hand let $M$ be a differentiable manifold with a contact structure. If we put
then we can find four tensors \( \varphi_j^i, \xi^i, \eta_i \) and \( g_{ji} \) so that they define an \( (\varphi, \xi, \eta, g) \)-structure. Such a structure is called a contact metric structure [5].

In an almost contact metric space there are four tensor fields \( N_{ji}^h, N_j^i, N_{ji} \) and \( N_j \) which are the analogue of the Nijenhuis tensor in an almost complex structure [5].

A contact metric space with \( N_{ji}=0 \) or \( N_{ji}^h=0 \) is called a K-contact metric space or a normal contact metric space respectively. Of course a normal contact metric space is a K-contact metric space and a K-contact metric space is a contact metric space [6]. In the following we consider a notation \( \eta^i \) instead of \( \xi^i \).

A K-contact metric space in which the Ricci tensor takes the form

\[
R_{ji} = a g_{ji} + b \eta_i \eta_j;
\]

is called a K-contact \( \eta \)-Einstein space, where \( a \) and \( b \) become constant \((n>3)\), and

\[
a + b = n - 1, \quad R = an + b
\]

hold good [7], [6].

Let \( R_{kji}^h \) be the Riemannian curvature tensor and put

\[
H_{ji} = \varphi^{kh} R_{kji}^h, \quad \text{then} \quad H_{ji} = - \frac{1}{2} \varphi^{kh} R_{kji}^h.
\]

In a contact metric space, \( \varphi_{ji} \) is a skew symmetric closed tensor and

\[
\nabla_j \varphi_j^i = (n-1) \eta_j,
\]

holds good, where \( \nabla_i \) denotes the covariant differentiation with respect to the Riemannian connection.

In a K-contact metric space the following identities are valid [6]:

\[
\nabla_j \eta_i = \varphi_{ji},
\]

\[
\nabla_k \varphi_{ji} + R_{rkji} \eta^r = 0,
\]

\[
R_{kji} \eta_l^k \eta^j = 0, \quad H_i \eta^j = 0,
\]

\[
R_{kji} \eta^k \eta^h = g_{ji} - \eta_j \eta_i,
\]

\[
R_{ii} \eta^j = (n-1) \eta_i.
\]

In a normal contact metric space
(1.18) \[ \nabla_k \phi_{ji} = \eta_j g_{ki} - \eta_i g_{kj}, \]

(1.19) \[ \eta_j R_{kji} = \eta_k g_{ji} - \eta_j g_{ki}, \]

(1.20) \[ \phi_j R_{ji} = H_{ji} + (n-2) \phi_{ji}, \]

and also (1.13), (1.17) hold good [6].

In a normal contact or K-contact metric space a vector \( \sigma \) is called an infinitesimal CL-transformation if it satisfies

(1.21) \[ \mathcal{L}_v \left( \eta_{ji} \right) = \rho_j \phi^h_i + \rho_i \phi^h_j + \alpha (\eta_i \phi^h_j + \eta_j \phi^h_i), \]

where \( \mathcal{L}_v \) is the operator of Lie derivative and \( \{ \eta_{ji} \} \) is Riemannian connection, \( \rho_i \) is a vector field and \( \alpha \) is a certain scalar [1], [3]. Contracting \( h \) and \( j \) in (1.21), we see that \( \rho_i \) is a gradient.

In a K-contact metric space an infinitesimal CL-transformation hold good the following relations [4].

(1.22) \[ \mathcal{L}_v R_{ji} = (1-n) \nabla_j \rho_i + 2 \alpha (\eta_j \eta_i - g_{ji}) + \eta_j \phi_i + \nabla_j \alpha + \eta_i \phi_j + \nabla_i \alpha, \]

(1.23) \[ \eta_j \mathcal{L}_v R_{kji} = \eta_i \nabla_k \rho_i - \eta_k \nabla_j \rho_i + \alpha (\eta_k g_{ji} - \eta_j g_{ki}). \]

Finally we shall prepare the following theorem which has been proved by H. Mizusawa and K. Takamatsu.

**Lemma.** In a normal contact metric space, if \( \sigma \) is an infinitesimal CL-transformation, then the following relation holds good [3].

(1.24) \[ \mathcal{L}_v g_{ji} = - \nabla_j \rho_i + \alpha (g_{ji} + \eta_j \eta_i). \]

\( \S 2. \) Infinitesimal CL-transformations in an \( \eta \)-Einstein normal contact metric space.

Let \( \sigma \) be an infinitesimal CL-transformation in a normal contact metric space. Substituting (1.21) and (1.24) into the identity

\[ \nabla_k \mathcal{L}_v g_{ji} = g_{ki} \mathcal{L}_v \left( \eta_{ji} \right) + g_{ji} \mathcal{L}_v \left( \eta_{ki} \right), \]

we get

(2.1) \[ R_{rij} \phi^h_j = (g_{ji} + \eta_j \eta_i) \nabla^h \alpha - (\delta^h_i + \eta^h_i) \nabla_j \alpha = \rho^h g_{ji} - \rho_j \phi^h_i, \]

\[ \mathcal{L}_v \left( \eta_{kj} \right) + 2 (\rho_k \delta^h_j + \rho_j \delta^h_k) = (\delta^h_k + \eta^h_k) \nabla_j \alpha + (\delta^h_j + \eta^h_j) \nabla_k \alpha - (g_{jk} + \eta_j \eta_k) \nabla^h \alpha \] [4].
Thus we have the following:

**Proposition 2.1.** Let \( v^i \) be an infinitesimal \( CL \)-transformation and \( \rho_i \) be its associated vector. If \( \alpha \) is constant then \( \rho_i \) is an infinitesimal projective transformation and conversely.

Now, we begin with some simple lemmas.

**Lemma 2.2.** In a \( K \)-contact metric space, for a vector field \( \rho_i \) if there exist \( \lambda \) and \( \mu \) such that

\[
\nabla_j \rho_i = \lambda g_{ji} + \mu \eta_j \eta_i,
\]

then we have \( \mu = 0 \) \([7]\).

**Proof.** Differentiating (2.2) covariantly and taking account of (1.13) we have

\[
\nabla_k \nabla_j \rho_i = g_{ji} \nabla_k \lambda + \eta_j \nabla_k \eta_i + \mu (\varphi_{kj} \eta_i + \varphi_{ki} \eta_j).
\]

Transvecting \( \varphi^{kj} \) to this and making use of (1.4), (1.5) and (1.11), we get

\[
H_{ij} \rho^j = -\varphi_{ij}^k \nabla_k \lambda + \mu (n-1) \eta_i.
\]

Transvecting the last equation with \( \eta_i \) and using of (1.15), we have \( \mu = 0 \). This completes the proof.

**Lemma 2.3.** In a \( K \)-contact metric space, for a vector field \( \rho_i \) if there exist scalars \( \lambda \) and \( \mu \) such that

\[
\nabla_j \rho_i = \lambda g_{ji} + \mu \eta_j \eta_i + c (\eta_j \rho_i \rho_r + \eta_i \rho_j \rho_r), \quad c = \text{constant},
\]

then we have \( \mu = c \lambda \).

**Proof.** Operating \( \nabla_k \) to (2.3), using of (1.13), we get

\[
\nabla_k \nabla_j \rho_i = g_{ji} \nabla_k \lambda + \eta_j \nabla_k \eta_i + \mu (\varphi_{kj} \eta_i + \varphi_{ki} \eta_j) + c (\varphi_{kj} \varphi_{ir} \rho^r + \varphi_{ki} \varphi_{jr} \rho^r + \eta_j \varphi_{ir} \nabla_k \rho^r + \eta_i \varphi_{jr} \nabla_k \rho^r + \eta_j \varphi_{ir} \nabla_k \rho^r + \eta_i \varphi_{jr} \nabla_k \rho^r).
\]

On the other hand, from (2.3) we have

\[
\varphi_{ir} \nabla_k \rho^r = \lambda \varphi_{ik} - c \eta_k \rho_i + c \eta_i \rho^r \eta_k \eta_i.
\]

Substituting (1.14) and the last equation into (2.4), we obtain

\[
\nabla_k \nabla_j \rho_i = g_{ji} \nabla_k \lambda + \eta_j \nabla_k \eta_i + \mu (\varphi_{kj} \eta_i + \varphi_{ki} \eta_j) + c (\varphi_{kj} \varphi_{ir} \rho^r + \varphi_{ki} \varphi_{jr} \rho^r - \eta_j \rho_i \rho^r \star \mathbf{R}_{ski}^r - \eta_i \rho_j \rho^r \star \mathbf{R}_{skj}^r).
\]
Note on infinitesimal conformal and CL-transformations

+η_i(λρ_{jk} - c η_k ρ_j + c η' ρ_r η_k η_j) + η_j(λρ_{ik} - c η_k ρ_i + c η' ρ_r η_k η_i).

Transvecting φ_k^i η^j to this and making use of (1.4), (1.5), (1.16) and (1.15), we get μ = cλ.

LEMMA 2.4. In an n (n ≥ 3) dimensional normal contact η-Einstein space (b ≠ 0), v^i is an infinitesimal CL-transformation then the following relation holds good.

\[ \nabla_j α = -\frac{b}{n}(ρ_j - η' ρ_j η_j). \]

PROOF. Contracting h and j in (2.1) we have

\[ -R_{jil} η^l + η_j η^l \nabla_i α - n \nabla_j α = (1-n)ρ_j. \]

Transvecting (2.6) with η^j and using of (1.17), we get η_j \nabla_i α = 0.

Thus (2.6) can be written as

\[ R_{jil} η^l + n \nabla_j α = (n-1)ρ_j. \]

Substituting (1.9) and (1.10) into the last equation, we obtain (2.5).

LEMMA 2.5. Let v^i be an infinitesimal CL-transformation in a normal contact η-Einstein space (n ≥ 3) with b ≠ 0, then v^i is a contact one.

PROOF. Taking of the Lie derivative of the both sides of (1.19) and substituting (1.23) into the equation thus obtained, we get

\[ R_{kji} η^h = g^{ji}_{v} η^h + η_{bk} η_{kj} - g^{k} η_{ji} - η_{j} g^{k}_{hi} - η_{j} \nabla_{b} η^l + \eta_{b} \nabla_{l} η^l + \alpha(η_{j} g^{j}_{ki} - η_{k} g^{j}_{ji}). \]

Transvecting (2.7) with φ_k^j, we have

\[ (φ_k^j R_{kji}^h + 2φ_i^h) η^h = 0. \]

Substituting (1.9), (1.11) and (1.20) into (2.8), we get

\[ η^i η_{ji} = η^i η_{ji}, \]

where we have put σ = η^i η^j v.

In an η-Einstein space with b ≠ 0, for any vector v^i we have

\[ θ R^j_{ij} = a η^i η_{ji} + b(η^i η^j + η^j η^i). \]

Substituting (1.22), (1.24), (2.5) and (2.9) into (2.10), we obtain
\begin{equation}
(2.11) \quad (1-n)\nabla_j\rho_i + 2\alpha(n\eta_j\eta_i - g_{ji}) + \frac{b}{n} (\eta_j\varphi_i^r\rho_r + \eta_i\varphi_j^r\rho_r) = \rho_i [-\nabla_j\rho_i + \alpha(g_{ji} + \eta_j\eta_i)] + 2b \sigma j\eta_i.
\end{equation}

**THEOREM 2.6.** In a normal contact $\eta$-Einstein space $(n \geq 3)$ with $a + 2 < 0$, $\psi$ be an infinitesimal CL-transformation with $\alpha$ constant, then $\psi$ is a concircular one.

**PROOF.** From Lemma 2.4, (2.11) can be written as
\begin{equation}
(2.12) \quad (1-n)\nabla_j\rho_i + 2\alpha(n\eta_j\eta_i - g_{ji}) = \rho_i [-\nabla_j\rho_i + \alpha(g_{ji} + \eta_j\eta_i)] + 2b \sigma j\eta_i.
\end{equation}

Applying Lemma 2.2 to (2.12), it follows that
\[-b\nabla_j\rho_i = (a+2)\alpha g_{ij},\]
which shows that the transformation is concircular.

**THEOREM 2.7.** In a compact normal contact $\eta$-Einstein space $(n \geq 3)$ with $a + 2 < 0$, let $\psi$ be an infinitesimal CL-transformation then $\psi$ is an infinitesimal isometry.

**PROOF.** Operating $\nabla_k$ to (2.5), we have
\[
\nabla_k\nabla_j\alpha = \frac{b}{n}(\nabla_k\rho_j - \varphi_j^k \rho_k - \eta_j\eta_k \nabla_k\rho - \eta_k\rho_k).
\]

Transvecting $g^{kj}$ to this and using of (1.4) we get
\begin{equation}
(2.13) \quad \Delta \alpha = \frac{b}{n}(\nabla^\rho\rho - \beta),
\end{equation}
where we put $\beta = \eta^\rho \nabla^\rho \rho^\rho$.

On the other hand, substituting (1.9), (1.10), (1.22) and (1.24) into the identity
\[
\xi R = g^{ji}(\xi R_{ji} + R_{ji} \xi g^{ji}),
\]
we obtain
\[0 = (1-n)\nabla^\rho\rho + (ag_{ji} + b\eta_j\eta_i) [\nabla^j\rho^i - \alpha(g^{ji} + \eta_j\eta_i)]\]
or
\begin{equation}
(2.14) \quad b(\nabla^\rho\rho - \beta) = -(a+2)(n-1)\alpha.
\end{equation}

Comparing with (2.13) and (2.14), it follows that
\begin{equation}
(2.15) \quad \Delta \alpha = -\frac{n-1}{n}(a+2)\alpha.
\end{equation}

(*) It is well known that $\psi + \frac{1}{2} \rho^i$ is an infinitesimal isometry [3].
Since \(a+2<0\), applying Green's theorem to (2.15), we have \(\alpha=0\) [9].

Last, applying Lemma 2.3 to (2.11), we get

\[
(2.16) \quad n \nabla_j \rho_i + \frac{(n-1)(a+2)}{b} \alpha g_{ji} - (\eta_i \rho_i') \rho_s + \eta_i \rho_j' \rho_s = 0.
\]

Thus, taking account of (2.5) and \(\alpha=0\), we have \(\nabla_j \rho_i = 0\).

Since our space is compact, we find \(\rho_i = 0\).

Hence \(\varphi^i\) is an infinitesimal isometry.

In an \(\eta\)-Einstein space it is known that if \(_v \mathcal{L} g_{ji} = 0\), then \(_v \mathcal{L} \eta_i = 0\) holds good [7].

By Theorem 2.7 and the identity

\[
\nabla_j \mathcal{L} \eta_i - \mathcal{L} \phi_{ji} = \eta_i \mathcal{L} \phi^{[j]}_{i},
\]

we have immediately the following [2]:

**COROLLARY 2.8.** In a compact normal contact \(\eta\)-Einstein space \((n>3)\) with \(a+2<0\), an infinitesimal CL-transformation is an automorphism.

§ 3. Curvature-preserving infinitesimal CL-transformation in a \(K\)-contact metric space.

M. Okumura has proved that, in a normal contact metric space any curvature-preserving infinitesimal transformation is necessary an infinitesimal isometry [8].

In this section we shall prove the following:

**THEOREM 3.1.** In a compact \(K\)-contact metric space, a curvature preserving infinitesimal CL-transformation is necessary an infinitesimal isometry.

**PROOF.** Transvecting \(g^i_{ji}\) to (1.22), we have \(\nabla_i \rho' = 0\). Therefore \(\rho_i = 0\).

Transvecting (1.22) with \(\eta^i \eta'_j\), we get

\[
(3.1) \quad (1-n) \eta^i \eta'_j \nabla_i \rho_s + 2\alpha (n-1) = 0.
\]

On the other hand, transvecting \(g^{ki} \eta'_j\) to (1.23) and taking account of \(\nabla' \rho_j = 0\), we obtain

\[
(3.2) \quad -\eta'^i \eta'_j \nabla_i \rho_s - \alpha (n-1) = 0.
\]

From (3.1) and (3.2), we find \(\alpha=0\), and hence \(_v \mathcal{L}^{[h]}_{ji} = 0\).

Since our space is compact, we have \(_v \mathcal{L} g_{ji} = 0\). This completes the proof.
In the proof of Theorem 3.1, we have immediately the following

**COROLLARY 3.2.** Let \( \nu^i \) be an infinitesimal CL-transformation and \( \rho_i \) be its associated vector in a \( K \)-contact metric space. In order that \( \nu^i \) be an infinitesimal curvature-preserving transformation, it is necessary and sufficient that \( \alpha \) be zero and \( \nabla_j \rho_i = 0 \).

§ 4. Infinitesimal \( \eta \)-conformal transformation.

In a contact metric space, we consider an infinitesimal transformation satisfying the following

\[
\mathcal{L}_\nu g_{ji} = \lambda (g_{ji} + \eta_j \eta_i),
\]

where \( \lambda \) is a scalar function. We shall call such a transformation an infinitesimal \( \eta \)-conformal one [2]. In the paper [2], H. Mizusawa has proved the following two theorems.

**THEOREM A.** In a \( K \)-contact metric space with constant scalar curvature \( R \equiv -(n-1) \), an infinitesimal \( \eta \)-conformal transformation with \( \lambda = \) constant is an infinitesimal isometry.

**THEOREM B.** In order that a transformation in a contact metric space be an infinitesimal isometry, it is necessary and sufficient that the transformation be infinitesimal \( \eta \)-conformal and infinitesimal affine at the same time.

Now, we shall prove the following:

**THEOREM.** 4.1. In a compact \( K \)-contact metric space \((n \geq 3)\) with constant scalar curvature \( R + (n-1) \leq 0 \), an infinitesimal \( \eta \)-conformal transformation is an infinitesimal isometry.

**PROOF.** Substituting (4.1) into the identity

\[
\mathcal{L}_\nu \{ \frac{\partial}{\partial \nu} g_{ji} \} = \frac{1}{2} \lambda^{h} (\nabla_j \mathcal{L}_\nu g_{ri} + \nabla_i \mathcal{L}_\nu g_{rj} - \nabla_r \mathcal{L}_\nu g_{ji}),
\]

we get

\[
(4.2) \quad \mathcal{L}_\nu \{ \frac{\partial}{\partial \nu} g_{ji} \} = \frac{1}{2} [\lambda (\nabla_j (\partial^h_i + \eta^h_i \eta_j) + \lambda_i (\nabla_j^h + \eta^h \eta_j) - \lambda^h (g_{ji} + \eta_j \eta_i)]
\]

\[
+ 2 \lambda (\phi_j^h \eta_i + \phi_i^h \eta_j), \quad \lambda_i = \partial_i \lambda.
\]

According to (1.14), (4.2) and the identity

\[
(4.3) \quad \mathcal{L}_\nu \{ R_{h ki} \} = \nabla_k \mathcal{L}_\nu \{ h \} - \nabla_j \mathcal{L}_\nu \{ k \},
\]

we have
\[ (4.4) \quad \mathcal{L}_{R_h} = \frac{1}{2} \left[ \eta_i (\lambda_i \varphi_j - \lambda_j \varphi_i) + \eta^j (\lambda_j \varphi_i - \lambda_i \varphi_j) + 2 \lambda \varphi_i \varphi_j \right] \\
\quad \quad \quad + \nabla_k \lambda (\varphi_j + \lambda_j \eta_k) - \nabla_j \lambda (\varphi_i + \lambda_i \eta_j) + \lambda_i (\varphi_i \eta_j - \varphi_j \eta_i) \\
\quad \quad \quad - \nabla_j \lambda (g_{ji} + \eta_j \eta_i) + \nabla_j \lambda (g_{ki} + \eta_k \eta_i) - \lambda_j (2 \eta_i \varphi_{kj} + \eta_j \varphi_{ki}) \\
\quad \quad \quad - \eta_i \varphi_{ji} + 2 \varphi_i (\lambda \eta_j - \lambda_j \eta_i) + 2 \lambda (2 \varphi_j \varphi_i - \varphi_i \varphi_j) \\
\quad \quad \quad + 2 \lambda (\eta_k \varphi_{ji} - \eta_j \varphi_{ki} - \eta_i \varphi_{ji})].
\]

Now taking the Lie derivative on both sides of (1.16), we obtain

\[ (4.5) \quad \eta^i \eta^j \mathcal{L}_{R_{hji}} + \eta^k \mathcal{L}_{R_{hji}} \eta^k + \eta^k R_{hji} \eta^k = \mathcal{L}_{g_{ij}} \eta_i \mathcal{L}_{g_{ji}} \eta_j - \eta_j \mathcal{L}_{g_{ij}} \eta_i.
\]

Substituting (4.1) and (4.4) into (4.5), transvecting \( g_{ji} \) to (4.5), and making use of (1.15) and (1.16) we get

\[ (4.6) \quad \frac{1}{2} [4 \beta - (n+1) \beta - 2 \nabla^r \lambda_r + 2 (n-1) \lambda] = \lambda (n+1) - 2 \eta_r \mathcal{L}_{\eta_r},
\]

where we put \( \beta = \eta^r \mathcal{L}_{\eta_r} \).

On the other hand, from (4.1) and the identity
\[ \frac{1}{2} \eta^r \mathcal{L}_{g_{rs}} = \eta^r \mathcal{L}_{\eta_r},
\]

we have

\[ (4.7) \quad \eta^r \mathcal{L}_{\eta_r} = \lambda.
\]

Making use of (4.6) and (4.7), we obtain

\[ (4.8) \quad 2 \nabla^r \lambda_r + (n-3) \beta = 0.
\]

According (4.1), (4.4) and the identity
\[ g_{ji} \mathcal{L}_{R_{ji}} + R_{ji} \mathcal{L}_{g_{ji}} = \mathcal{L} R = 0,
\]

we have

\[ (4.9) \quad -n \nabla^r \lambda_r + \beta - \lambda (R + n - 1) = 0.
\]

Substituting (4.9) into (4.8) to eliminate \( \beta \), we get

\[ (4.10) \quad \nabla^r \lambda_r = - \frac{(n-3)(R + n - 1)}{(n-1)(n-2)} \lambda, \quad (n > 3).
\]

Applying Green’s theorem to (4.10), we have \( \lambda = 0 \) if \( R + n - 1 \leq 0 \). This completes the proof.

We have also from theorem 4.1.

**Corollary 4.2.** In a compact \( K \)-contact \( \eta \)-Einstein space \( (n > 3) \) with \( a + 2 < 0 \),
an infinitesimal $\eta$-conformal transformation is an automorphism.

THEOREM 4.3. In order that a transformation in an Einstein (or compact) contact metric space be an infinitesimal isometry, it is necessary and sufficient that the transformation be infinitesimal $\eta$-conformal and infinitesimal CL-transformation at the same time.

PROOF. By theorem B, the necessity is evident. We shall prove the sufficiency. From (1.21) and (4.2), it follows that

$$
\rho_j \tilde{\rho}_i^h + \rho_i \tilde{\rho}_j^h + \alpha(\eta_j \varphi_i^h + \eta_i \varphi_j^h) = \frac{1}{2} [\lambda_i (\tilde{\varphi}_i^h + \eta_i^h \eta_i) + \lambda_i (\tilde{\varphi}_j^h + \eta_j^h \eta_j) - \lambda^h (g_{ji} + \eta_j \eta_i) + 2 \lambda (\varphi_j^h \eta_i + \varphi_i^h \eta_j)],
$$

Contracting (4.11) with respect to $j$ and $h$, we get $2\rho_i = \lambda_i$.

Next transvecting (4.11) with $\eta_h$, we obtain

$$
\eta_j \rho_i + \eta_i \rho_j = (\eta_i \varphi_j^h) (g_{ji} + \eta_j \eta_i),
$$

from which we have $\eta_i \varphi_j^h = 0$ and $\rho_i = 0$. From these and (4.11), we find $\lambda = \text{constant}$, $\alpha = \lambda$.

By the identity (4.3) and $\lambda = \text{constant}$ it follows that

$$
\mathcal{L} R_{ji} = \lambda \nabla_i (\varphi_j^h \eta_i + \varphi_i^h \eta_j),
$$

$$
\mathcal{L} R = g^{ji} \mathcal{L} R_{ji} + R_{ji} \mathcal{L} g^{ji} = - \lambda (R + R_{ji} \eta^i \eta^j).
$$

If we assume that the space be Einstein, we have

$$
\mathcal{L} R = - \frac{n+1}{n} \lambda R = 0,
$$

from which we get $\lambda = 0$. If the space be compact, from (4.1) we have

$$
\nabla^i v_i = \frac{n+1}{2} \lambda.
$$

By Green's theorem we have $\lambda = 0$. These complete the proof.

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