Iterative Solutions of the Dirichlet Problem
for $\Delta u = bu^{2n}$

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1. Introduction. We construct the solution of

$$\Delta u = bu^{2n} \tag{1}$$

in a given region, the value of $u$ on the boundary of the region being known. The given boundary values of $u$ and the constant $b$ are non-negative. The existence of the solution is demonstrated by the convergence of the iterative methods discussed.

The notation and terminology of this paper are based on [1]. Under the same conditions of [1], we can easily show that the solution $u$ is unique by an appeal to the maximum principle and its corollary in [1]. Three iterative methods belonging to the class of linear iterative methods

$$\Delta u_{i+1} = bu_{i}^{n}(cu_{i}^{n} + (1-c)u_{i+1}^{n})$$

are investigated.

2. Newton's Methods. The problem of this paper is to determine the nonnegative solution $u$ of the integral equation

$$u(s) + b\int G(s, t)u^{2n}(t)dt - h(s) = 0 \tag{2}$$

which is equivalent to (1). Identifying $P(u)$ with the left-hand side of (2), one has

$$P'(u)v(s) = v(s) + 2nb\int G(s, t)u^{2n-1}v(t)dt,$$

and

$$P''(u)vw(s) = 2n(2n-1)b\int G(s, t)u^{2n-2}v(t)w(t)dt.$$
From (2), we have
\[ P(u_0) = b \int G(s, t) u^{2n}(t) \, dt. \]
Since the maximum value of \( h \) is unity, the maximum of \( f \),
\[ P(u_0) \leq b \int G(s, t) \, dt \leq bG \] [1].

From the definition of the Fréchet derivative,
\[ P'(u_0)u = u(s) + 2nb \int G(s, t) h^{2n-1}(t) u(t) \, dt. \]

Using the bound of the Green's function [1],
\[ \max 2nb \int G(s, t) h^{2n-1}(t) u(t) \, dt \leq 2nbG_1 \max u(s). \]

If \( 2nbG_1 < 1 \), the inverse of the operator \( P'(u) \) exists and its norm is no greater than
\[ B = 1/(1 - 2nbG_1) \] [3].

We obtain the following inequality
\[ \| [P'(u_0)]^{-1} P(u_0) \| \leq \| [P'(u_0)]^{-1} \| \cdot \| P(u_0) \| \leq B \cdot bG_1 = \eta. \]

A bound for \( P''(u) \) with \( u \) in a certain neighborhood of \( u_0 \) is needed. However, \( P''(u) \) has the general bound
\[ K = 2n(2n-1)bG_1. \]

Finally, we need
\[ \frac{1}{2} \geq BK \eta = \frac{2n(2n-1)bG_1^2}{(1 - 2nbG_1)^2}. \] (3)

If \( b \) is sufficiently small, this inequality will be satisfied. Thus for the present problem the Kantorovich Theorem [1] asserts that if \( b \) is so small that inequality (3) is satisfied, then the sequence \( u_i \) of solutions to
\[ u_0(s) = h(s) \]
\[ u_{i+1}(s) + 2nb \int G(s, t) u_i^{2n-1}(t) u_{i+1}(t) \, dt \]
= \( h(s) + (2n-1)b \int G(s, t) u_i^{2n}(t) \, dt \) \( (i=0, 1, 2, \ldots \) converges to the solution \( u \) of (2), that solution being unique.

3. The Natural iteration. The successive iterates \( u_i \) are solutions of
\[ u_0 = h \]
\[ \Delta u_{i+1} = bu_i^{2n} \quad \text{in } R \]
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\[ u_i = f \quad \text{on } S(i = 0, 1, 2, \ldots) \]

or equivalently

\[ u_{i+1}(s) = h(s) - b \int G(s, t) u_i^{2n}(t) dt. \]

$h$ is positive in $R$ \[{1}\]. Since $\int G(s, t) u_0^{2n}(t) dt$ is positive and zero on $S$, for small enough $b$, $u_1(s)$ will also be nonnegative in $R: 0 \leq u_1$.

The solution $u$ is no greater than $h$. For $\Delta (u - h) = bu^{2n} \geq 0$, so that, by the maximum principle, $u - h \leq 0$.

If $b$ is so small that $0 \leq u_1$, then

\[ 0 \leq u_1 \leq u_2 \leq \ldots \leq u_{i+1} \leq \ldots \leq u_i \leq \ldots \leq u_1 \leq u_0 \]

by further application of the maximum principle.

The convergence of the sequence $u_i$ is established as follows:

\[ |u_{i+1} - u_i| = b |\int G(s, t) [u_i^{2n} - u_{i-1}^{2n}] (t) dt| \]

\[ = b |\int G(s, t) (u_i^n + u_{i-1}^n)(u_i^n - u_{i-1}^n)(t) dt| \]

\[ \leq b \max |u_i^n - u_{i-1}^n| \cdot |\int G(s, t) (u_i^n + u_{i-1}^n) dt| \]

\[ \leq 2b \max |u_i^n - u_{i-1}^n| \cdot |\int G dt| \]

\[ \leq 2bG_1 \max |u_i^n - u_{i-1}^n| \]

\[ \leq 2bG_1 \max |u_i^{n+1} - u_i - u_{i-1} + \ldots + u_{i-1}^{n-1} + u_i^{n-1}| \cdot |u_i - u_i^{n-1}| \]

\[ \leq 2nbG_1 \max |u_i^n - u_i - u_{i-1}^n|. \]

If $2nbG_1 < 1$, the sequence converges absolutely and uniformly.

4. The iteration for arbitrary positive values of $b$. For this method the sequence of functions $u_i$ is taken to be:

\[ u_0 = h \]

\[ \Delta u_{i+1} = bu_i^{2n} \quad \text{in } R \]

\[ u_i = f \quad \text{on } S \quad (i = 0, 1, 2, \ldots) \]

or equivalently

\[ u_{i+1}(s) = h(s) - b \int G(s, t) u_i^n(t) u_{i+1}^n(t) dt. \]  \[ (4) \]

For this sequence one may assert that the $u_i$ are positive in $R$ except
perhaps for a set of measure zero on which \( u_i'' = 0 \) (i.e., \( u_i = 0 \)) [1].

To demonstrate the convergence of the iteration, it is shown below that
\[ \| v_{i+1} \| \leq r \| v_i \| \quad (0 < r < 1), \]
where \( v_i = u_{i+1}'' - u_i'' \).

Thus
\[ \| v_{i+1} \| \leq r^{i+1} \| v_0 \|. \]

Here the norm is that of real \( L_a \)-space:
\[ \| p \| = \left( \int p^2(t) \, dt \right)^{1/2}. \]

\[ \lim_{i \to \infty} u_i'' = u \]
exists as the sum of the convergent in the series
\[ u_0 + \sum_{i=1}^{\infty} v_i. \]

From (4),
\[ u_{i+2} - u_{i+1} = -b \int G(s, t) [u_{i+1}'' u_{i+2}''(t) - u_i'' u_{i+1}''(t)] \, dt \]
\[ = -b \int G(s, t) u_{i+1}''(v_{i+1} + v_i)(t) \, dt. \]

Therefore
\[ v_{i+1} = -b [u_i'' + \ldots + u_i''] \int G(s, t) u_{i+1}''(v_{i+1} + v_i)(t) \, dt \quad (5). \]

Let \( \{ \varphi_j \} \) be the set of normalized characteristic solutions and the numbers \( \lambda_j \), the corresponding characteristic values of
\[ \int G(s, t) u_{i+1}''(t) \varphi_j(t) \, dt = \lambda_j \varphi_j(s). \]

The existence and completeness of the \( \varphi_j \) and \( \lambda_j \) under certain continuity conditions on \( u_i'' \) (i.e., \( u_{i+1}'' \)) is demonstrated in [2].

The corresponding differential equation for \( \varphi_j \) is
\[ \lambda_j \varphi_j = -u_{i+1}'' \varphi_j \text{ in } R \]
\[ \varphi_j = 0 \text{ on } S. \]

Here we have
\[ \lambda_j \int \varphi_j \dot{\varphi}_j \, dt = -\int u_{i+1}'' \varphi_j^2 \, dt. \]

By Green's theorem
\[ -\lambda_j \int \varphi_j^2 \, dt = -\int u_{i+1}'' \varphi_j^2 \, dt. \]
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so that, $\lambda_j \geq 0$.

A bound on the $\lambda_j$ is obtained from the integral form:

$$\lambda_j |\varphi_j(s)| = |\int G(s, t)u_{i+1}^*(t)\varphi_j(t)dt|$$

$$\leq \max |\varphi_j| \cdot \max |u_{i+1}^*| \cdot \max |\int G(s, t)dt|$$

$$= G_1 \max |\varphi_j|.$$

It follows that $\lambda_j \leq G_1$.

By the completeness of the $\varphi_j$, there exist constants $a_{i,j}$ and $a_{i+1,j}$ for which

$$v_i = \sum_{j=1}^{\infty} a_{i,j} \varphi_j,$$

$$v_{i+1} = \sum_{j=1}^{\infty} a_{i+1,j} \varphi_j.$$

Inserting the series forms for $v_i$ and $v_{i+1}$ into (5) gives

$$\sum_{j=1}^{\infty} \{a_{i+1,j} [1 + b \lambda_j (u_{i+2}^{* -1} + \cdots + u_{i+1}^{* -1})] + a_{i,j} b \lambda_j (u_{i+2}^{* -1} + \cdots + u_{i+1}^{* -1}) \} \varphi_j = 0.$$

From the orthogonality of the $\varphi_j$,

$$a_{i+1,j} = -a_{i,j} b \lambda_j (u_{i+2}^{* -1} + \cdots + u_{i+1}^{* -1}) 1 + b \lambda_j (u_{i+2}^{* -1} + \cdots + u_{i+1}^{* -1})^{-1}.$$

Since $0 \leq \lambda_j \leq G_1$ and $u_{i+2}^{* -1} + \cdots + u_{i+1}^{* -1} \leq n$,

$$0 \leq \frac{b \lambda_j (u_{i+2}^{* -1} + \cdots + u_{i+1}^{* -1})}{1 + b \lambda_j (u_{i+2}^{* -1} + \cdots + u_{i+1}^{* -1})^{-1}} \leq \frac{nbG_1}{1 + nbG_1} = r < 1.$$

So that

$$|a_{i+1,j}| \leq r |a_{i,j}|.$$

By Parseval's theorem, we have

$$||v_{i+1}|| \leq r ||v_i||.$$

Here $r$ is independent of $i$. Hence the sequence $u_i$ converges in the mean. Therefore $u_i$ converges in the mean, as was to be shown.

References


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