ON CHAINS OF MEIJER-LAPLACE TRANSFORM OF TWO VARIABLES

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1. Introduction

The well known Laplace transform of two variables [2, p.39]

\[ F(p, q) = pq \int_0^\infty e^{-px-xy} f(x, y) \, dx \, dy, \quad R(p, q) > 0 \]  

has been generalized by the author in the form [3]

\[ F(p, q) = pq \int_0^\infty G_{m, m+1}^n (px/(a_1, \ldots, a_m)) G_{n, n+1}^{n+1, 0} (qy/(b_1, \ldots, b_n)) \times f(x, y) \, dx \, dy, \quad R(p, q) > 0. \]  

The Meijer-Laplace transform (1.2) will be denoted symbolically as \( F(p, q) \) = \( G(f(x, y)) \) whereas the Laplace transform (1.1) as \( F(p, q) = f(x, y) \). When \( b_j = 0, j = 1, 2, \ldots, m-1; d_i = 0, i = 1, 2, \ldots, n-1 \) and

(a) \( b_m = a_{m+1} = d_n = c_{n+1} = 0 \), using \( G_{0, 1}^0 (z/\bar{0}) = e^{-x} \), (1.2) reduces to (1.1);

(b) \( b_m = -m-k, a_m = m-k, a_{m+1} = -m-k; d_n = -m_1-k_1, c_n = m_1-k_1 \).

\[ c_{n+1} = -m_1-k_1, \] (1.2) reduces to

\[ F(p, q) = pq \int_0^\infty e^{-\frac{1}{2}px - \frac{1}{2}qy} (px)^{-k-\frac{1}{2}} (qy)^{-k_1-\frac{1}{2}} W_{k+\frac{1}{2}, m} (px) \times W_{k_1+\frac{1}{2}, m_1} (qy) f(x, y) \, dx \, dy, \quad R(p, q) > 0 \]

and is known as Meijer transform of two variables [4, p.83];

(c) \( a_m = 2m, b_m = \frac{1}{2} - m-k, a_{m+1} = 0; c_n = 2m_1, d_n = \frac{1}{2} - m_1-k_1, c_{n+1} = 0 \).

(1.2) reduces to

\[ F(p, q) = pq \int_0^\infty e^{-\frac{1}{2}px - \frac{1}{2}qy} (px)^{m-\frac{1}{2}} (qy)^{m_1-\frac{1}{2}} W_{k, m} (px) W_{k_1, m_1} (qy) \]

\[ (g) \] For brevity, the symbol \( \int_0^\infty \) denotes \( \int_0^\infty \), the symbol \( R(p, q) > 0 \) denotes \( R(p) > 0, R(q) > 0 \) and \( (g_r) \) denotes the set of parameters \( g_1, g_2, \ldots, g_r \).
which we shall call as Varma transform of two variables [5].

In this paper we have obtained a chain of Meijer-Laplace transform of two variables which yield interesting results in other transforms to which it reduces.

In what follows the symbol \( \Delta(n,a) \) denotes the set of parameters \( \frac{a}{n}, \frac{a+1}{n}, \ldots, \frac{a+n-1}{n} \), where \( n \) is a positive integer and the symbol \( \Delta(n,a_i) \) denotes the set of parameters \( \Delta(n,a_1), \Delta(n,a_2), \ldots, \Delta(n,a_r) \).

2. We shall require the following results which follow from the results given by Saxena [6, p. 401]:

\[
(2.1) \quad \int_{0}^{\infty} t^{\alpha-1} G_{\alpha,\beta}^{\delta,\gamma} \left( \frac{p}{t} \right) G_{\nu,\delta}^{\gamma,\zeta} \left( \frac{a}{t} \right) dt
\]

where \( \alpha \) is a positive integer.

\[
(2.2) \quad G_{2\alpha,0}^{2\alpha,0} \left( \frac{p_s}{2\alpha} / \frac{2\alpha}{4(2\alpha-1-2\alpha)} \right) = 2^{3\alpha-2} \pi^{1/2} e^{-p_s}
\]

where \( \alpha \) is a positive integer.

3. THEOREM. If

\[
(3.1) \quad F(p,q) = G(f_1(s,t)),
\]

\[
(3.2) \quad (pq)^{\frac{1}{2}} f_1 \left( \frac{1}{p}, \frac{1}{q} \right) = G(f_2(s,t)),
\]

\[
(3.3) \quad \frac{\pi}{4} \left( \frac{1}{pq} \right)^{\frac{1}{2}} f_2 \left( \frac{1}{4p^2}, \frac{1}{4q^2} \right) = G(f_3(s,t)),
\]

\[
(3.4) \quad \frac{\pi}{4} \left( \frac{1}{pq} \right)^{\frac{1}{2}} f_3 \left( \frac{1}{4p^2}, \frac{1}{4q^2} \right) = G(f_4(s,t)),
\]

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and

\[
\left\langle 3.5 \right\rangle \quad \frac{\pi}{4} \left( \frac{1}{pq} \right)^{\frac{1}{2}} f_{-1} \left( \frac{1}{4p^2}, \frac{1}{4q^2} \right) = G(f_s(s, t)),
\]

then

\[
\left\langle 3.6 \right\rangle \quad F(p, q) = 2r^{2r-2} \pi^{2-2r} \prod_2 \left( \alpha^{a_{m+1}+c_{n+1}-\Sigma^m_{m+1} b_i - \Sigma^m_{n+1} d_i} \right)
\]

\[
\times p q \int_0^\infty \int G_{2a, m, 2c, m+1} \left( \frac{p s^{2a}}{2^a}, \frac{q t^{2c}}{2^c} \right) \text{d} \left( \frac{1, a_{m+1} - \frac{2a-1}{2}}{c_{m+1}}, d \left( \frac{1, a_{m+1} - \frac{2a-1}{2}}{c_{m+1}}, d \left( \frac{1, a_{m+1} - \frac{2a-1}{2}}{c_{m+1}} \right) \right) \right)
\]

\[
\times s^{2a} t^{2c} f \left( \frac{s^2}{4}, \frac{t^2}{4} \right) \text{dsdt},
\]

provided \( R(p, q) > 0 \), the Meijer-Laplace transform of \( |f_k(s, t)| \) for \( k = 1, 2, \ldots, r \) all exist, and the integrals involved are absolutely convergent. Here \( \alpha = 2r-2 \) and \( \prod_2 \) indicates the product of the factors within bracket for \( r = 2 \) to any integral value of \( r \).

**Proof.** Substituting the value of \( f_1(s, t) \) from (3.2) in (3.1), interchanging
the order of double integration which is permissible due to absolute convergence of the integrals, using \([1, p.209, (9)]\) and evaluating the later double integral with the help of (2.1), replacing \( s \) by \( \frac{s^2}{4} \) and \( t \) by \( \frac{t^2}{4} \), we get

\[
\left\langle 3.7 \right\rangle \quad F(p, q) = 2r^{2r-2} \pi^{2-2r} \prod_2 \left( \alpha^{a_{m+1}+c_{n+1}-\Sigma^m_{m+1} b_i - \Sigma^m_{n+1} d_i} \right)
\]

\[
\times p q \int_0^\infty \int G_{2a, m, 2c, m+1} \left( \frac{p s^{2a}}{2^a}, \frac{q t^{2c}}{2^c} \right) \text{d} \left( \frac{1, a_{m+1} - \frac{2a-1}{2}}{c_{m+1}}, d \left( \frac{1, a_{m+1} - \frac{2a-1}{2}}{c_{m+1}} \right) \right)
\]

\[
\times s^{2a} t^{2c} f \left( \frac{s^2}{4}, \frac{t^2}{4} \right) \text{dsdt}.
\]

From (3.3) putting the value of \( f_2(s, t) \) in (3.7) and proceeding as above, we get

\[
\left\langle 3.8 \right\rangle \quad F(p, q) = 2r^{2r-2} \pi^{2-2r} \prod_2 \left( \alpha^{a_{m+1}+c_{n+1}-\Sigma^m_{m+1} b_i - \Sigma^m_{n+1} d_i} \right)
\]

\[
\times \int_0^\infty \int G_{2a, m, 2c, m+1} \left( \frac{p s^{2a}}{2^a}, \frac{q t^{2c}}{2^c} \right) \text{d} \left( \frac{1, a_{m+1} - \frac{2a-1}{2}}{c_{m+1}}, d \left( \frac{1, a_{m+1} - \frac{2a-1}{2}}{c_{m+1}} \right) \right)
\]

\[
\times s^{2a} t^{2c} f \left( \frac{s^2}{4}, \frac{t^2}{4} \right) \text{dsdt}.
\]
\[ \times \mathcal{G}_{4n, 4(n+1)} \left( \frac{qt^4}{4^4}, \left( \left( 2, c_i + d_i, -\frac{3}{2} \right), \left( \left( 1, c_i, d_i, -\frac{1}{2} \right) \right) \right) \right) \]

\[ \times (st)^4 f_3 \left( \frac{s^2}{4}, \frac{t^2}{4} \right) ds dt \]

Repeating this process successively with correspondences (3.4), ..., we arrive at the result.

**COROLLARY.** Taking \( b_j = 0, j = 1, 2, \ldots, m \), \( a_{m+1} = 0 \); \( d_i = 0 \), \( i = 1, 2, \ldots, n \), \( c_{n+1} = 0 \), simplifying and then using (2.2), we get the following chain in Laplace transform of two variables.

If

\[ F(p, q) \approx f_1(s, t), \]

\[ (pq)^{\frac{1}{2}} f_1 \left( \frac{1}{p}, \frac{1}{q} \right) \approx f_2(s, t), \]

\[ \frac{\pi}{4} \left( \frac{1}{pq} \right)^{\frac{1}{2}} f_2 \left( \frac{1}{4 p^2}, \frac{1}{4 q^2} \right) \approx f_3(s, t), \]

\[ \frac{\pi}{4} \left( \frac{1}{pq} \right)^{\frac{1}{2}} f_3 \left( \frac{1}{4 p^2}, \frac{1}{4 q^2} \right) \approx f_4(s, t), \]

\[ \ldots \]

and

\[ \frac{\pi}{4} \left( \frac{1}{pq} \right)^{\frac{1}{2}} f_{r-1} \left( \frac{1}{4 p^2}, \frac{1}{4 q^2} \right) \approx f_r(s, t), \]

then

\[ F(p^{2r-1}, q^{2r-1}) \approx \frac{\pi st}{4} f_r \left( \frac{s^2}{4}, \frac{t^2}{4} \right), \]

provided \( R(p, q) > 0 \), and the Laplace transform of \( |f_i(s, t)| \) and

\[ \left| st f_i \left( \frac{s^2}{4}, \frac{t^2}{4} \right) \right| \text{ for } i = 2, 3, \ldots, r \text{ all exist.} \]

Putting \( b_j = 0, j = 1, \ldots, m-1 \), \( b = -m-k \), \( a = -m-k \), \( a_{m+1} = -m-k \), \( d_i = 0 \), \( i = 1, \ldots, n-1 \), \( d_n = -m_1-k_1 \), \( c_n = m_1-k_1 \), \( c_{n+1} = -m_1-k_1 \) in (3.1), (3.2), ..., we get a corresponding chain of Meijer transform of two variables.

Setting \( b_j = 0, j = 1, 2, \ldots, m-1 \), \( b = \frac{1}{2} - m-k \), \( a = 2m \), \( a_{m+1} = 0 \); \( d_i = 0 \), \( i = 1, \ldots, n-1 \), \( d_n = \frac{1}{2} - m_1-k_1 \), \( c_n = 2m_1 \), \( C_{n+1} = 0 \) in (3.1), (3.2), ..., we get a chain of Varma transform of two variables.
The author is extremely thankful to Dr. R.K. Saxena for his help and guidance in the preparation of this paper.

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