NOTES ON CLOSED HYPERSURFACES IN A RIEMANNIAN SPACE
WITH CERTAIN DIFFERENTIAL EQUATION

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1. Introduction

Let $M$ be an $n$-dimensional orientable Riemannian manifold covered by a system of coordinate neighbourhoods $(t^h)$ and $g_{ji}$, $\nabla$, $K_{jih}$, $K_{ji}$, and $K$, the positive definite fundamental tensor, the operator of covariant differentiation with respect to Christoffel symbols $\{h_i^j\}$ formed with $g_{ji}$, the curvature tensor, the Ricci tensor, and the curvature scalar of $M$ respectively, where here and in the following the indices $h, i, j, \ldots$ run over the range $1, 2, \ldots, n$. Recently K. Yano [1] proved

THEOREM. Let $M$ be an orientable Riemannian manifold of dimension $n$ which admits a non-constant scalar field $v$ such that

$$\nabla_j \nabla_i v = f(v)g_{ji},$$

where $f$ is a differentiable function of $v$ and $S$ a closed orientable hypersurface in $M$ such that

(i) its first mean curvature is constant,

(ii) $[K_{ji} + (n-1)f(v)g_{ji}]\mathbf{C}^i\mathbf{C}^j > 0$ on $S$, where $\mathbf{C}^i$ is the unit normal to $S$,

(iii) the inner product $\mathbf{C}^i \nabla_i v$ has fixed sign on $S$.

Then every point of $S$ is umbilical.

To obtain a generalization of above theorem, we assume in this paper the existence of a non-constant scalar function which satisfies similar partial differential equation. While under an arbitrary conformal transformation $g_{jk} = v^2 g_{jk}$ any geodesic circle will be transformed into a geodesic circle if and only if the function $v$ satisfies the relation

$$\nabla_j \nabla_i v = a g_{ji} + v_i v_j,$$

where

$$v_j = \frac{\partial \log v}{\partial x^j}$$

and such a conformal transformation will be called concircular [4].
2. Formulas in $M$ admitting any Scalar Field

We consider a closed orientable hypersurface $S$ in a Riemannian manifold $M$ whose local parametric equations are

$$\xi^h = \xi^h(\eta^a),$$

$\eta^a$ being parameters on $S$, where here and in the following the indices $a, b, c, \cdots$ run over the range $1, 2, \cdots, n-1$.

If we put

$$B^h_b = \partial^h_b \xi^h, \quad \partial^h_b = \partial / \eta^b,$$

then $B^h_b$ are $n-1$ linearly independent vectors tangent to $S$ and the first fundamental tensor of $S$ is given by

$$g_{cb} = g_{jiB^h_c B^h_b}.$$

We assume that $n-1$ vectors $B^h_1, B^h_2, \cdots, B^h_{n-1}$ give the positive orientation on $S$ and we denote by $C^h$ the unit normal vector to $S$ such that

$$B^h_1, B^h_2, \cdots, B^h_{n-1}, C^h$$

give the positive orientation in $M$.

Denoting by $\nabla_c$ the operator of van der Waerden-Bortolotti covariant differentiation along $S$ we have the following equations of Gauss and of Weingarten:

$$(2.1) \quad \nabla_c B^h_b = h_{cb} C^h,$$

$$(2.2) \quad \nabla_c C^h = -h_{c}^a B^h_a,$$

where $h_{cb}$ is the second fundamental tensor of $S$ and $h_{c}^a = h_{cb} g^{ba}$. We also obtain the equations of Gauss and those of Codazzi in the form

$$K_{ijk} B^k_d B^j_c B^i_b = K_{deba} - (h_{da} h_{cb} - h_{db} h_{ca}),$$

where $K_{deba}$ is the curvature tensor of the hypersurface $S$. From the equations of Codazzi, we have, by a transvection with $g^{cb}$,

$$(2.3) \quad K_{kh} B^k_d C^h = \nabla_d h^c_c - \nabla_c h^c_d.$$

We now assume that the Riemannian manifold $M$ admits a non-constant scalar field $\nu$ such that

$$(2.4) \quad \nabla_i \nu = f(\nu) g_{ji} + \nu_j v_i, \quad v_i = \nabla_i \nu$$

where $f(\nu)$ is a differentiable function of $\nu$.

The condition (2.4) is a formal generalization of a concircular transformat-
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ion in a Riemannian space. We shall call such a transformation a \textit{f-concircular transformation}.

We put

\begin{equation}
(2.5) \quad v^h = B^h_a v_a + \alpha C^h
\end{equation}

on the hypersurface \(S\). From (2.4) we obtain by transvection with \(B^i_j B^j_i\)

\begin{equation}
(2.6) \quad \nabla_c v_b = f(v) g_{cb} + \alpha h_{cb} + v_b v_c
\end{equation}

from which

\begin{equation}
(2.7) \quad \Delta v = (n-1) f(v) + \alpha k^c_c + v^a v_a,
\end{equation}

where \(\Delta\) is the Laplacian operator on \(S\): \(\Delta = g^{cb} \nabla_c \nabla_b\).

From (2.4), we also obtain by transvection with \(B^i_j C^j_i\)

\begin{equation}
(2.8) \quad \nabla_b \alpha = -h_b^a v_a + \alpha v_b
\end{equation}

On the other hand, substituting (2.4) into the Ricci identity

\begin{equation}
\nabla_k \nabla_j v_i - \nabla_j \nabla_k v_i = -K^k_{hji} v_h,
\end{equation}

we find that

\[-K^k_{hji} v_h = f'(v) (v_k g_{ji} - v_j g_{hi}) + f(v) (g_{ki} v_j - g_{ij} v_k),\]

from which

\[K^i_{hji} v_i = -(n-1) (f'(v) - f(v)) v_i,\]

and consequently

\[K^i_{hji} C^j_i = -(n-1) \alpha (f'(v) - f(v)),\]

which can also be written as

\[K^i_{hji} (B^j_i v^c + \alpha C^j_i C^j_i) = -(n-1) \alpha (f'(v) - f(v)),\]

or, by virtue of (2.3),

\[(\nabla_c h^b_c - \nabla_v h^b_v) v^c + \alpha K^i_{hji} C^j_i C^j_i = -(n-1) \alpha (f'(v) - f(v)),\]

that is

\begin{equation}
(2.9) \quad \alpha K^i_{hji} C^j_i C^j_i + (n-1) (f'(v) - f(v)) \alpha + v^c \nabla_c h^b_c - \nabla_v (h^b_c v^c) + h_{cb} v^b v^c + f(v) h^b_v + \alpha h_c^b h^c_b = 0
\end{equation}

by virtue of (2.6).

We now assume that the hypersurface \(S\) is closed and the first mean curvature of \(S\) is constant.

Applying Green's theorem to (2.7) and (2.9), we obtain
(2.10) \( (n-1) \int_S f(v) dS + \int_S \alpha h_c \, dS + \int_S \nu \nu_a dS = 0 \)
and
(2.11) \( \int_S [\alpha K_{ji} C_i^j + (n-1) \alpha (f'(v) - f(v)) + f(v) h_b^b + \alpha h_c^b h_b^c + h_c^b v^b] dS = 0 \)
respectively, where \( dS \) denotes the surface element of \( S \).
Eliminating \( \int_S f(v) ds \) from these two equations, we find that
(2.12) \( \int_S [\alpha (K_{ji} + (n-1) \alpha (f'(v) - f(v)) + \alpha ^2 (h_c^b - \frac{1}{n-1} h_c^t g_c^b) (h_b^c - \frac{1}{n-1} h_b^t g_b^c) + (h_c^b - \frac{1}{n-1} h_c^t g_c^b) v^c v^b] dS = 0 \)
On the other hand, from (2.6) and (2.8) we have
\[
\nabla_c \nabla_b \alpha = - (\nabla h_{ab} ) v^c - h_{ab} (\alpha h_c^a + v_c v^a + f(v) \delta_c^a ) + \delta_c^a v^b + \alpha (\alpha h_{cb} + v_c v_b + f(v) g_{cb}),
\]
from which
(2.13) \( \Delta \alpha = - \nabla_c (C^a h_a^c) - h_{ab} \alpha \delta_b^c + 2 \alpha h_c^c + 2 \alpha v a + f(v) (n-1) \alpha . \)
Applying Green's theorem to (2.13) we find that
(2.14) \( \int_S [h_{cb} v^c \alpha^2 v^c - 2 \alpha v a - (n-1) \alpha f(v)] dS = 0. \)
From (2.10), (2.11) and (2.14) we have
(2.15) \( \int_S \alpha \left[ (K_{ji} + (n-1) f'(v) g_{ji}) C_i^j C^i (h_{cb} - \frac{1}{n-1} h_{cb} g_{cb}) (h_b^c - \frac{1}{n-1} h_b^c g_b^c) + \int_S \right. \left. 2 \alpha \frac{n}{n-1} h_c^c v^c v_a + \alpha^2 h_c^c \right] dS = 0. \)

3. Results

From (2.12) and (2.15) we have immediately the following

THEOREM 1. Let \( M \) be an orientable Riemannian manifold of dimension \( n \) which admits a proper \( f \)-concircular transformation and \( S \) a closed orientable hypersurface in \( M \) such that
(i) its first mean curvature is constant,
(ii) \( K_{ji} + (n-1) (f'(v) - f(v)) g_{ji} ) C_i^j C^i \equiv 0 \) on \( S \), where \( C^i \) is the unit normal to \( S \),
(iii) \( (h_{cb} - \frac{1}{n-1} h_{cb} g_{cb}) v^c v^b \), \( C^i \nabla_i v \) have the same fixed sign on \( S \),
or (i) and (ii) \( [K_{ji} + (n-1) f'(v) g_{ji} ] C_i^j C^i \equiv 0 \) on \( S \)
(iii) \( C^i \nabla_i v \equiv \frac{1}{2(n-1)} h_a^a \geq 0 \)
on \( S \).
Then every points of \( S \) is umbilical.
THEOREM 2. Let $M$ be an orientable Riemannian manifold of dimension $n$ which admits a proper $f$-concircular transformation and $S$ a closed orientable hypersurface in $M$ such that

(i) its first mean curvature is constant,

(ii) $C^i \nabla_i \nu$ is positive on $S$,

(iii) $\{K_{ij} + [(n-1)(f'(\nu)-f(\nu))-\frac{1}{4\alpha^2}v^a v_a] g_{ji}\} C^i C^j \equiv 0$ on $S$.

Then $v^h$ is normal to $S$.

PROOF. From (2.10) and (2.11) we get

$$\int_S \alpha \left[ K_{ji} C^i C^j + (n-1)(f'(\nu)-f(\nu)) + h^c_b h^d_c - \frac{1}{n-1} h^c_b h^d_c - \frac{1}{\alpha(n-1)} h^c_b v^a v_a 
+ \frac{1}{\alpha} h^c_{ab} v^a v^b \right] dS = 0,$$

by virtue of $C^i \nabla_i \nu = \alpha$.

or

$$\int_S \alpha \left[ (n-1)(f'(\nu)-f(\nu)) + \frac{1}{4\alpha^2} v^a v_a] g_{ji}\right] C^i C^j
+ \left( h^c_{ab} - \frac{1}{n-1} h^c_{ab} g^c_{ab} + \frac{1}{2\alpha} v^c v^b \right) \left( h^c_b - \frac{1}{n-1} h^c_b g^c_b + \frac{1}{2\alpha} v^c v^b \right) dS = 0.$$

Therefore $h^c_{ab} - \frac{1}{n-1} h^c_{ab} g^c_{ab} + \frac{1}{2\alpha} v^c v^b = 0$. Hence $v^h = 0$.

These complete the proof.

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REFERENCES