A CORRECT SYSTEM OF AXIOMS FOR A SYMMETRIC GENERALIZED TOPOLOGICAL GROUP

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In this paper we introduce the concept of a symmetric generalized topological group and show its relationship to the concept of a symmetric generalized uniform space (introduced by the author in [2]).

Throughout this paper \((G, \cdot)\) will denote a group with identity \(e\). \(\mathcal{T}\) will denote a topology on \(G\), and \(\mathcal{N}\) will denote an open base at \(e\). \(A^{-1} = \{a^{-1} \mid a \in A\}\), \(AB = \{ab \mid a \in A, b \in B\}\) where \(A\) and \(B\) are subsets of \(G\).

(1.1) DEFINITION. \((G, \cdot, \mathcal{T})\) is a symmetric generalized topological group iff the following axioms are satisfied:

\[(A.1)\] For every \(x \in G\) \(\{xN \mid N \in \mathcal{N}\}\) is an open base at \(x\).

\[(A.2)\] For every \(N \in \mathcal{N}\) \(N = N^{-1}\).

(1.2) REMARK. If we require that the mapping \(f: (x, y) \to xy\) of \((G \times G)\) onto \(G\) be continuous in each variable separately, then \(\{xN \mid N \in \mathcal{N}\}\) and \(\{Nx \mid N \in \mathcal{N}\}\) are bases at \(x\) for every \(x \in G\). If we require that the mapping \(g: x \to x^{-1}\) of \(G\) onto \(G\) be continuous, then for every \(N \in \mathcal{N}\) \(N^{-1} \in \mathcal{N}\). This latter fact implies that for every \(N \in \mathcal{N}\) \((N \cap N^{-1}) \in \mathcal{N}\). But \((N \cap N^{-1})^{-1} = (N \cap N^{-1})\).

(1.3) REMARK. It is easily shown that if \(F\) is closed, \(P\) is open, and \(A\) is an arbitrary subset of \(G\) and if \(x\) is an arbitrary point in \(G\), then \(xF, P^{-1}\), are closed and \(xP, P^{-1}\), and \(AP\) are open subsets of \(G\) where \((G, \cdot, \mathcal{T})\) is a symmetric generalized topological group.

(1.4) THEOREM. If \((G, \cdot, \mathcal{T})\) is a symmetric generalized topological group, then \(A = \{AN \mid N \in \mathcal{N}\}\).

PROOF. Let \(y \in A\) and \(N \in \mathcal{N}\). Then \(yN^{-1} \cap A \neq \emptyset\); so that \(y \in AN\). Conversely, suppose \(y \in AN\) for every \(N \in \mathcal{N}\). Then \(y \in AN^{-1}\) for every \(N \in \mathcal{N}\); consequently, \(yN \cap A \neq \emptyset\) for every \(N \in \mathcal{N}\); so that \(y \in A\).

(1.5) THEOREM. Let \((G, \cdot, \mathcal{T})\) be a symmetric generalized topological group. Let \(F\) be a closed and \(C\) a compact subset of \(G\) such that \(F \cap C = \emptyset\). Then there exists \(N \in \mathcal{N}\) such that \(FN \cap CN = \emptyset\).
PROOF. Let \( M \in \mathcal{N} \). Let \( F_M = FMM^{-1} \). Then by Theorem (1.4) \( F_M = \cap \{ FM \cdot M^{-1} W \mid W \in \mathcal{N} \} \). Hence \( F_M \cap C = \emptyset \) for each \( M \in \mathcal{N} \); so that \( \{ G - F_M \mid M \in \mathcal{N} \} \) is an open covering of \( C \). Hence there is a finite subfamily \( F_M, (1 \leq i \leq n) \) such that

\[
\left( \bigcap_{i=1}^{n} F_M \right) \cap C = \emptyset.
\]

Let \( N = \cap M_i \). It is easily shown that

\[
NN^{-1} = \cap M_i N^{-1} \subseteq \cap M_i M_i^{-1}
\]

so that

\[
FNN^{-1} \subseteq \cap F_M M_i^{-1}.
\]

By taking closures we see that

\[
FNN^{-1} \subseteq F_N \subseteq \cap F_M.
\]

Hence \( FNN^{-1} \cap C = \emptyset \); so that \( FN \cap CN = \emptyset \).

We now investigate the relationship between symmetric generalized topological groups and symmetric generalized uniform spaces.

(1.6) THEOREM. Let \( (G, \cdot, \mathcal{T}) \) be a symmetric generalized topological group. For each \( N \in \mathcal{N} \) let \( U_N = \{(x, y) \mid x^{-1}y \in N\} \). Let \( \mathcal{B} \) be the collection of all \( U_N \). Then \( \mathcal{B} \) is a base for a symmetric generalized uniformity, \( \mathcal{U}(G) \), on \( G \) such that \( \mathcal{T}(\mathcal{U}(G)) = \mathcal{T} \).

NOTE. \( AN = \bigcup \{ zN \mid z \in A \} \); so that by (A.1) \( AN \) is open. But by hypothesis there exists \( b \in AN \cap B \). Since \( AN \) is open, \( b \) is an interior point of \( AN \); consequently, by (A.1) there exists \( W \in \mathcal{N} \) such that \( bW \subseteq AN \).

PROOF of THEOREM (1.6). Clearly, to show \( \mathcal{B} \) is a base for some symmetric generalized uniformity \( \mathcal{U} \) on \( G \) it is sufficient to show that for every \( N \in \mathcal{N} \) and for all subsets \( A, B \) of \( G \), if \( U_M [A] \cap B \neq \emptyset \) for every \( M \in \mathcal{N} \), then there exists \( b \in B \) and there exists \( W \in \mathcal{N} \) such that \( U_W [b] \subseteq U_N [A] \). But since we have that \( U_N [A] = \cup \{ zN \mid z \in A \} = AN \) for all \( N \in \mathcal{N} \) and for each subset \( A \) of \( G \), this is an immediate consequence of the note above. It is clear that \( \mathcal{T}(\mathcal{U}(G)) = \mathcal{T} \).

(1.8) COROLLARY. If \( (G, \cdot, \mathcal{T}) \) is a symmetric generalized topological group and \( \mathcal{N} \) has a least element, say \( N_0 \), then \( N_0^2 \subseteq N \) for every \( N \in \mathcal{N} \).

PROOF. Clearly, for every \( U \in \mathcal{U}(G) \) we have that \( U_{N_0} \subseteq U \). Consequently, by lemma (2.32) in [2] \( U_{N_0} \circ U_{N_0} \subseteq U \) for every \( U \in \mathcal{U}(G) \). Hence if \( (x, y) \in U_{N_0} \).
and \( (y, z) \in U_{N_0} \), then \( (x, z) \in U_{N} \) for every \( N \in \mathcal{N} \). That is to say for every \( N \in \mathcal{N} \) if \( x^{-1}y \in N_0 \) and \( y^{-1}z \in N_0 \), then \( x^{-1}z \in N_0 \). Let \( p \in N_0 \) and \( q \in N_0 \). Then \( p^{-1} \) is in \( N_0 \); so that \( P^{-1} \in \) is in \( N_0 \) and \( \varepsilon^{-1}q \) is in \( N_0 \). Hence \( pq \in N \). Thus \( N^2 \subset N \).

(1.9) Theorem. If \( (G, \cdot, \mathcal{T}) \) is a locally compact symmetric generalized topological group, then \( \mathcal{U}(G) \) is complete.

Proof. Let \( \mathcal{F} \) be any filter in \( G \) that is weakly Cauchy with respect to \( \mathcal{U}(G) \). Since \( (G, \cdot, \mathcal{T}) \) is locally compact, there exists a compact neighborhood \( N \in \mathcal{N} \), and since \( \mathcal{T} \) is weakly Cauchy with respect to \( \mathcal{U}(G) \), there exists an \( x_0 \in G \) such that \( U_N(x_0) = x_0N \in \mathcal{F} \). By (A.1) it is easily shown that \( xN \) is compact. We now let \( \mathcal{B} = \{E \subset F \cap x_0N \text{ for some } F \in \mathcal{F} \} \). It is easily shown that \( \mathcal{B} \) is a base for a filter \( \mathcal{F}_1 \) in \( x_0N \); but since \( x_0N \) is compact, \( \mathcal{F}_1 \) has a cluster point \( x_1 \in x_0N \); which clearly is a cluster point for \( \mathcal{F} \). Hence \( (G, \mathcal{U}(G)) \) is complete.

(1.10) Theorem. If \( (G, \cdot, \mathcal{T}) \) is a locally compact, \( T_2 \), symmetric generalized topological group, then \( (G, \cdot, \mathcal{T}) \) is a topological group.

For a proof of this rather deep result the reader is referred to [1].

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References