On the Weak Primitivity of Matrix Rings

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1. Introduction. Let $R$ be a ring. If there exists a faithful modular maximal right ideal $I$ of $R$, $R$ is called primitive [1]. Let $R_n$ be the ring of $n \times n$ matrices, where $n$ is a positive integer. E.C. Posner [5] proved that $R_n$ is primitive if and only if $R$ is primitive. The author defines a ring $R$ to be weakly primitive if and only if there exists a faithful almost maximal right ideal $I$ of $R$. The purpose of this note is to prove that $R_n$ is weakly primitive if and only if $R$ is weakly primitive, to mention that $R_n$ is weakly transitive if and only if $R$ is weakly transitive.

In section 2, some necessary fundamental concepts including almost maximal right ideals, notations, and lemmas will be stated. In section 3, it will be proved that the weak primitivity of $R$ implies that of $R_n$. In section 4, conversely it will be proved that the weak primitivity of $R_n$ implies that of $R$. Finally it will be concluded that the weak primitivity of $R_n$ is equivalent to that of $R$, with applications this fact as corollaries.

2. Preliminaries. Let $I$ be a right ideal of a ring $R$. Then $I$ is called faithful if and only if $R/I$ is a faithful (right) $R$-module, i.e.,

$$[I : R] = \{r \in R | Rr \subseteq I\} = \{0\}.$$ 

Evidently for $a \in R$,

$$[I : a] = \{r \in R | ar \in I\}$$

is a right ideal of $R$. It is to see that

$$N(I) = \{r \in R | rI \subseteq I\}$$

is the largest subring of $R$ in which $I$ is two-sided ideal. This subring $N(I)$ is called the normalizer of the right ideal $I$.

DEFINITION 1. A proper right ideal $I$ of a ring $R$ is called almost maximal if and only if

(1) for right ideals $J_1, J_2$, $J_1 \cap J_2 = I$ implies $J_1 = I$ or $J_2 = I$, that is, $I$ is meet irreducible,

(2) $a R$ and $[I : a] \supseteq I$ implies $a \in I$,

(3) for a right ideal $J$ such that $J \supseteq I$, $N(I) \cap J \supseteq I$ holds and also $a \in R$ such
that \([J:a]\supset I, [J:a]\supset I\) holds.

A ring \(R\) is a right order in a ring \(L\) with a unity if and only if \(R\) is a subring of \(L\), each regular (not zero divisor) element of \(L\) is a unit, and if \(q \in L\) there exist a regular element \(r \in R\) and an element \(s \in R\) such that, in \(L\), \(q=sr^{-1}\).

Let \(M\) be a \((right)\) \(R\)-module. Then \(M\) is called uniform if and only if every pair of non-zero submodules of \(M\) has a non-zero intersection. A weakly transitive ring of linear transformations is defined as follow:

DEFINITION 2. Let \(V\) be a left vector space over a division ring \(D\) and let \(R\) be a ring of linear transformations of \(V\). Consider \(V\) as a \((right)\) \(R\)-module. Then \(R\) is weakly transitive provided that there is a right order \(K\) in \(D\) and a \((K,R)\)-submodule \(M\) of \(V\) such that \(M\) is uniform as a \(R\)-module, \(DM=V\), and such that if \(\{m_i\}_{i=1}^n\) is a finite \(D\)-linearly independent subset of \(M\) and if \(\{y_i\}_{i=1}^n\) is a sequence from \(M\), then there exists \(r \in R, k \in K, k \neq 0\) such that \(m_ir=ky_i, 1 \leq i \leq n\).

Useful lemmas will be stated whose proofs are to be found in [3] and [4] respectively.

LEMMA 1. Let \(I\) be an almost maximal right ideal of a ring \(R\). Then \(M=R-I\) is a uniform \((right)\) \(R\)-module with centralizer \(K=N(I)/I\), and \(M\) has the extended centralizer \(D[2]\) in which \(K\) is a right order, and \(M\) can be extended to a \((D,R)\)-module \(V\) such that \(DM=V\) and such that \(R\) induces a weakly transitive ring of linear transformations of the vector space \(V\) over \(D\).

From the weak transitivity of this lemma, we draw a corollary which will be used later.

COROLLARY. Let \(I\) be an almost maximal right ideal of a ring \(R\). If \(0 \neq t \in I\), then there exists an element \(b\) in \(R\) such that \(tb \in N(I)\) and \(tb \notin I\).

Proof. Since \(te I, l \neq 0\) in \(M=R-I\). For any non-zero \(b\) of \(K=N(I)/I\), there exists a non-zero \(b\) in \(K\) and an element \(b\) in \(R\) such that \(tb=hb\), which is naturally not zero and belongs to \(K=N(I)/I\), according to the weak transitivity of LEMMA 1.

Hence \(tb \in N(I)\) but \(tb \notin I\).

LEMMA 2. If \(I\) is an almost maximal right ideal of a ring \(R\) and \(M=R-I\), then for each \(m \in M, m \neq 0\) the right ideal \(m'=\{r \in R|mr=0\}\) is almost maximal.

3. Weak Primitivity of \(R^*\). As stated in Introduction, we proceed to show that the weak primitivity of \(R\) implies that of \(R^*\).
PROPOSITION 1. If there exists a faithful right ideal $I$ of $R$, then there exists a faithful right ideal $I^*$ of $R^*$. 

Proof. Suppose $I$ is a right ideal of $R$ such that $[I : R] = (0)$. Let 

$$ I^* = \{(r_{ij}) \in R_n | r_{kj} \in I, \ 1 \leq j \leq n \} $$

for a fixed $k$. (Throughout this section $k$ is a fixed positive integer.) Then $I^*$ is a right ideal of $R^*$. If $(s_{ij}) \in [I^* : R_n]$, then $R_n(s_{ij}) \subseteq I^*$, that is, $rE_{kp}(s_{ij}) \in I^*$ for all $r$ of $R$, $1 \leq j, p \leq n$. Therefore $rE_{kp} \in I$ for all $r$ of $R$, $1 \leq j, p \leq n$, namely, $R_nE_{kp} \subseteq I$. Hence $s_{pq} = 0$, $1 \leq j, p \leq n$, because $[I : R] = (0)$. This means $(s_{ij}) = (0)$ and proves that $[I^* : R_n] = (0)$, that is, $I^*$ is faithful.

PROPOSITION 2. Let $I$ be an almost maximal right ideal of $R$, and let 

$$ I^* = \{(r_{ij}) \in R_n | r_{kj} \in I, \ 1 \leq j \leq n \}. $$

If $J^* \subsetneq I^*$ for a right ideal $J^*$ of $R_n$, then there exist two matrices $\delta \in R_n$, $\beta \in I^*$ such that $\beta \delta e N(I^*) \cap J^*$, $\beta \delta e I^*$, that is, $N(I^*) \cap J^* \subsetneq I^*$, and $\beta \delta$ is the form in which every entity is 0 except the $k$-th column.

Proof. Let $\beta = (b_{ij}) \in J^* \cap I^*$. Then there exists $b_{e} \in R$ such that $b_{e} \in I$ for some $e$. $1 \leq e \leq n$. From Corollary, it follows that there exists an element $d \in R$ such that $b_{d} e N(I)$. $b_{d} \in I$. Let $\beta = dE_{dk}$. Then $\beta \delta e J^*$. 

$$ \beta \delta = \sum b_{i} d E_{ik} $$

and this matrix has the form in which every entity is 0 except the $k$-th column.

If $\alpha = (a_{ij}) \in I^*$, then $(\beta \delta) \alpha e I^*$ since the entities of the $k$-th row of $(\beta \delta) \alpha$: 

$$ b_{e} d a_{e}, \ b_{e} d a_{2}, ..., b_{e} d a_{n} $$

are in $I$. Hence $\beta \delta e N(I^*)$. and $\beta \delta e I^*$ since $b_{i} d \in I$.

PROPOSITION 3. Let $I$ and $I^*$ be the same as in Proposition 2, and let $J^*_s$, $s = 1, 2$, be right ideals of $R_n$ such that $J^*_s \supsetneq I^*$. Then $J^*_1 \cap J^*_2 \supsetneq I^*$.

Proof. Let $J_s$, $s = 1, 2$, be the set of $(k, k)$-entities of all matrices which belong to $J^*_s$, $s = 1, 2$. Since it is clear that $I \subseteq J_1$, we show that $I \subseteq J_s$. Let $\beta \in J^*_s$, $\beta \notin I^*$. Then, by the argument in the proof of Proposition 2, there exists a matrix $\delta \in R_n$ for each $s$ such that the $(k, k)$-entity of $\beta \delta$, belongs to $J_s$, but not to $I$. Hence $J^*_s \subseteq I$, $s = 1, 2$. And consequently $J^*_1 \cap J^*_2 \supsetneq I$ by the meet-irreducibility of $I$. Let $t \in J^*_1 \cap J^*_2$, but $t \notin I$. Then we can choose an element $b \in R$ such that $tb \in N(I)$ but $tb \notin I$ by Corollary. On the other hand, there exist matrices $\beta_s \in J^*_s$, $s = 1, 2$, each of which is such that the $(k, k)$-entity is equal to $t$. And we can suppose, in particular, its every entity is 0 except the $k$-th row since $\beta_s = (t_{ij}) \in J^*_s$ can be written as 

$$ \sum t_{ij} E_{ij} = \sum t_{ij} E_{kj} + \sum t_{ij} E_{ij} $$

where $\sum t_{ij} E_{ij} \in I^* \supsetneq I^*$. Hence $\beta_s(bE_{kk}) = \beta_s(bE_{kk}) = tbE_{kk} \in J^*_1 \cap J^*_2$. but $tbE_{kk} \notin I^*$. This proves that $J^*_1 \cap J^*_2 \supsetneq I^*$.
PROPOSITION 4. Let \( I \) and \( I^* \) be the same as in proposition 2. If \( \beta = (b_{ij}) \in R_n \) and \( [I^*: \beta \supset I^*] \), then \( \beta \in I^* \).

Proof. First we show that \( I \subseteq [I : b_{ij}] \), \( 1 \leq j \leq n \). Let \( a \in I \). Then \( \beta a \in I^* \), since \( aE_j \in I^* \), \( 1 \leq j \leq n \). Hence \( b_{ij}a \in I \), i.e., \( I \subseteq [I : b_{ij}] \), \( 1 \leq j \leq n \). From the hypothesis \([I^*: \beta \supset I^*] \), we can choose \( \tau = (t_{ij}) \in I^* \) such that \( \beta \tau \in I^* \). We assume that \( t_{kk} \). If \( j \neq k \), then \( \beta t_{kj}E_j \in I^* \) since \( t_{kk}E_j \in I^* \). Therefore \( t_{kj} \in [I : b_{kj}] \). Hence \( [I : b_{kj}] \supset I \). If \( j = k \), then \( \beta (\sum t_{kj}E_j) \in I^* \), and \( b_{kk} \in I \), since \( \beta \in I^* \).

\[ \beta \tau = \beta (\sum t_{kj}E_j) + \beta (\sum t_{ij}E_j), \]

where \( \sum t_{ij}E_j \in I^* \) and \( \beta \sum t_{ij}E_j \in I^* \). Therefore \( t_{kj} \in [I : b_{kj}] \). Now it follows that \( [I : b_{kj}] \supset I, 1 \leq j \leq n \). Hence \( b_{kj} \in I \), \( 1 \leq j \leq n \). This proves that \( \beta \in I^* \).

PROPOSITION 5. Let \( I \) and \( I^* \) be the same as in Proposition 2.

If \( J^* \supset I^* \) for a right ideal \( J^* \) of \( R_n \) and if \( \beta J^* \cap J^* \subseteq I^* \), then \( \beta J^* \cap J^* \subseteq I^* \).

Proof. Suppose \( \beta = (b_{ij}) \in R_n \) such that \( \beta I^* \subseteq J^* \). Assume \( b_{kk} \) is in \( I \) and let \( \gamma = \sum c_{ij}E_{ij}, \gamma \in I^* \). Then \( \beta \gamma \in I^* \cap J^* \), since every entity of the \( k \)-th row of \( \beta \gamma \) is in \( I \). Hence \( [J^*: \beta \supset I^*] \). Now assume \( b_{kk} \in I \). And let \( J \) be the set of \( (k, k) \)-entities of all matrices in \( J^* \) whose entities are 0 except the \( k \)-th column. Then evidently \( J \) is a right ideal of \( R \) and \( J \supset I \). This inclusion is proper, i.e., \( J \supset I \), since there exists a matrix \( (r_{ij}) \in J^* \cap N(I^*) \), \( (r_{ij}) \notin I^* \) such that \( r_{ij} = 0 \) for \( j \neq k \), by Proposition 2. Also \( b_{kk} I \subseteq J \), i.e., \( [J : b_{kk}] \supset I \) since for every element \( a \in I \), \( aE_{kk} \in I^* \), \( \beta aE_{kk} \in J^* \), and the \( (k, k) \)-entity \( b_{kk} \) of \( \beta aE_{kk} \) is in \( J \). Hence \( [J : b_{kk}] \supset I \), by the almost maximality of \( I \). Now we can choose \( c \in [J : b_{kk}] \), \( c \notin I \). Let \( \gamma = cE_{kk} \). Then \( \beta \gamma \in J^*, \) i.e., \( \gamma \in [J^*: \beta \supset I^*] \) since \( b_{kk} \in J \). But \( \gamma \notin I^* \). Hence again \( [J^*: \beta \supset I^*] \).

It is clear that from Proposition 1~Proposition 5, the following theorem holds.

THEOREM 1. Let \( R \) be a ring and let \( R_n \) be the ring of \( n \times n \) matrices over \( R \), where \( n \) is a positive integer. Then the weak primitivity of \( R \) implies that of \( R_n \).

4. Weak Primitivity of \( R \). Before we proceed to show that the weak primitivity of \( R \) implies that of \( R_n \), we establish some properties.

LEMMA 3. Let \( J^* \) be an almost maximal right ideal of \( R_n \). Then there exists a matrix \( \gamma = b_{pp}E_{pp} \) for some \( b_{pp} \in R \) and for some \( p \), \( 1 \leq p \leq n \), such that \( I^* = [J^*: \gamma] \) is an almost maximal right ideal of \( R_n \), and \( \gamma \in N(I^*), \gamma \notin I^* \).

Proof. Suppose \( rE_{pq} \in J^* \) for some \( r \in R \) and for some \( p, q \), \( 1 \leq p, q \leq n \). Then there exists a matrix \( \beta \in R_n \) such that \( rE_{pq} \beta \in N(J^*), rE_{pq} \beta \notin J^* \) by Corollary. Let \( \beta = rE_{pq} \beta_1 \) then \( \beta \) is a matrix whose every entity is 0 except the \( p \)-th row. Now let \( \beta = (b_{ij}) \) and \( \gamma = b_{pp}E_{pp} \). Then \( \gamma \beta = \beta \gamma \). Since \( N(J^*)/J^* \) is an integral domain, \( \beta^2 \in J^*, \beta \in J^* \), thus \( \gamma \in J^* \). Evidently \( I^* = \{ \rho \in R_n | \rho \in J^* \} = [J^*: \gamma] \) is a right ideal of
$R_\alpha$ and its almost maximality follows from Lemma 2.

To show $r \in N(I^*)$, suppose $\rho \in I^*$. Then $\gamma \rho \in J^*$ and $\beta(\gamma \rho)$ is in $J^*$ since $\beta \in N(J^*)$. But $\gamma \rho = \beta r$, and $\beta r \rho = \gamma \rho = \gamma (\gamma \rho)$. Hence $\gamma \rho \in I^*$, and $\gamma \in N(J^*)$.

Now suppose $\gamma \in I^*$. Then $\beta \gamma = \gamma \gamma J^*$, thus $\gamma \in [J^*; \beta]$. But $\gamma J^*$ and $\beta \in N(J^*)$. Hence $[J^*; \beta] \supseteq J^*$ and $\beta \in J^*$, which is a contradiction. Therefore $\gamma \in I^*$.

**Lemma 4.** Let $\gamma = b_{pp}E_{pp}$ and $I^*$ be the same as in Lemma 3, and let $I$ be the set of $(p, p)$-entities of all matrices in $I^*$. Then $I$ is a right ideal of $R$ and $b_{pp} \in I$.

**Proof.** It is clear that $I$ is a right ideal of $R$. Now assume $b_{pp} \in I$. Then we can choose $\rho = (r_{ij}) \in I^*$ with $r_{pp} = b_{pp}$. Now $\rho \gamma \in I^*$ and

$$\rho \gamma - b_{pp}b_{pp}E_{pp} = \sum r_{ij}b_{pp}E_{ij}. $$

But $\sum r_{ij}b_{pp}E_{ij} \in I^*$ since $\gamma (\sum r_{ij}b_{pp}E_{ij}) = (0) \in J^*$. Hence $\gamma = b_{pp}b_{pp}E_{pp} \in I^*$. Therefore $\gamma \in [I^*; \gamma]$. But $[I^*; \gamma] \supseteq I^*$ since $\gamma \in N(I^*)$. This is impossible because of Lemma 3.

**Proposition 6.** Let $I$ be the same as in Lemma 4. Then, for a right ideal $J$ of $R$ such that $J \supseteq I$, $J \cap N(I) = I$ holds.

**Proof.** Let $\gamma$ and $I^*$ be the same as in Lemma 4. Choose an element $\gamma$ such that $\gamma \in J, \gamma \in I$ and define $\rho = rE_{pp}$. Then $\rho \in I^*$. Therefore there exists $\tau = (t_{ij}) \in R_\alpha$ such that $\rho \tau \in N(I^*)$ but $\rho \tau \in I^*$ by Corollary.

First, we show that $\rho \tau \in N(I)$. Let $a \in I$. Then there exists $s \in I^*$ such that $s_{pp} = a$. Now $\rho \tau \in I^*$ and

$$\gamma \sigma = (b_{pp}E_{pp})(\sum t_{ij}E_{ij}) = (\sum \sigma_{pp}E_{pp}).$$

Hence $\gamma (\sum s_{pp}E_{pp}) \in J^*$ and $\sigma_{pp} = \sum s_{pp}E_{pp} \in I^*$. Therefore, since $(\rho \tau) \sigma_{pp} \in I^*$ and

$$(\rho \tau) \sigma_{pp} = (rE_{pp})(\sum t_{ij}E_{ij})(\sum t_{pp}E_{pp})$$

$$= \sum r_{pp}s_{pp}E_{pp},$$

$\sum r_{pp}s_{pp}E_{pp} \in I^*$. Hence, in particular, $rt_{pp}s_{pp} = rt_{pp}a \in I$. It follows that $rt_{pp}I \subseteq I$, that is, $rt_{pp} \in N(I)$.

Next, to prove that $rt_{pp} \in I$, suppose $\mu = (m_{ij}) \in I^*$ such that $m_{pp} = rt_{pp}$. Now $\mu \gamma \in I^*$ and

$$\mu \gamma = (\sum m_{ij}E_{ij})(b_{pp}E_{pp}) = \sum m_{ij}b_{pp}E_{pp}$$

$$= \sum m_{ij}b_{pp}E_{pp} \in I^*. $$

$\sum m_{ij}b_{pp}E_{pp} \in I^*$. Therefore

$$m_{pp}b_{pp}E_{pp} = rt_{pp}b_{pp}E_{pp}$$

is in $I^*$. But

$$(\rho \tau) \gamma = (rE_{pp})(\sum t_{ij}E_{ij})(b_{pp}E_{pp})$$

$$= rt_{pp}b_{pp}E_{pp}. $$
Hence \((\rho \tau) \gamma \in I^\ast\), that is, \(\gamma \in [I^\ast : \rho \tau]\). But \(\gamma \notin I^\ast\) by Lemma 3. Therefore \([I^\ast : \rho \tau] \supset I^\ast\), which is a contradiction to the fact that \(\rho \tau \notin I^\ast\). Hence \(r \rho \tau \in I\).

Now, we have proved that \(J \cap N(I) \supset I\).

REMARK: In the proof of Proposition 6, we notice that if \(r \notin I\) then there exists an element \(t \in R\) such that \(r t \notin I\).

LEMMA 5. Let \(I^\ast\) and \(I\) be the same as in Lemma 4. And let
\[
I^{**} = \{(r_{ij}) \in R_n | r_{ij} \in I, 1 \leq j \leq n\}.
\]
Then \(I^\ast = I^{**}\).

Proof. To show that, \(I^\ast \subseteq I^{**}\), suppose \(\rho = (r_{ij}) \in I^\ast\). Now \(\rho (r E_{ij}) \in I^\ast, 1 \leq j \leq n\), and
\[
\rho (r E_{ij}) = \sum_i r_{ij} r E_{ij}.
\]
Hence \(r_{ij} \in I\) for all \(r\) of \(I\), \(1 \leq j \leq n\). By the above-mentioned Remark, \(r_{ij} \in I\), \(1 \leq j \leq n\).

This proves that \(I^\ast \subseteq I^{**}\).

Conversely, suppose that \(\rho = (r_{ij}) \in I^{**}\), that is,
\[
r_{mm}, r_{m1}, \ldots, r_{rn}
\]
are in \(I\), but \(\rho = (r_{ij}) \notin I^\ast\). Then there exists \(r \in R\) such that \(\rho r \in N(I^\ast), \rho \tau \notin I^\ast\) by Corollary. Let \(\rho r = \sigma (s_{ij})\). Then \(s_{pp} \notin I\); for, if \(\mu = (m_{ij}) \in I^\ast\) with \(m_{pp} = s_{pp}\), then
\[
m_{pp} b_{pp} E_{pp} = s_{pp} b_{pp} E_{pp}
\]
is in \(I^\ast\) where \(\gamma = b_{pp} E_{pp}\), by the argument in the proof of Proposition 6, consequently \((\rho r) \gamma = \sigma \gamma \in I^\ast\) by the argument in the proof of Lemma 4, leading to a contradiction as in the proof of Proposition 6. On the other hand, evidently every entity of the \(p\)-th row of \(\rho \tau = \sigma\) is in \(I\). This is again a contradiction. Therefore we conclude that \(\rho = (r_{ij}) \in I^\ast\), namely, \(I^{**} \subseteq I^\ast\).

The above result \(I^\ast = I^{**}\) will be used conveniently in the proof of the following proposition.

PROPOSITION 7. Let \(I\) be the same as in Lemma 4. Then
(1) \(I\) is meet-irreducible.
(2) \([J:a] \supset I\) for a right ideal \(J\) such that \(J \supset I\) and for an element \(a\) of \(R\) such that \([J:a] \supset I\).
(3) \(a \in R\) and \([I:a] \supset I\) implies \(a \in I\).

Proof. (1) Assume \(J_1\) and \(J_2\) are right ideals of \(R\) and \(J_1 \supset I\), \(J_2 \supset I\). Let
\[
J_i^\ast = \{(r_{ij}) \in R_n | r_{ij} \in J_i, 1 \leq j \leq n\}, i = 1, 2.
\]
Then \(J_1^\ast\) and \(J_2^\ast\) are right ideals of \(R\). And \(J_2^\ast \supset I^\ast\), since \(I^\ast = I^{**}, s = 1, 2\). Hence \(J_1^\ast \cap J_2^\ast \supset I^\ast\), from which it follows easily that \(J_1 \cap J_2 \supset I\), since \(I^\ast = I^{**}\).

(2) If \(J \supset I\) and \(a \in R\) such that \(a I \supset J\), let \(\sigma = a E_{pp}\). And let
\[
J^\ast = \{(r_{ij}) \in R_n | r_{ij} \in J, 1 \leq j \leq n\}.
\]
Then $aI^* \subseteq J^*$ since $I^* = I^{**}$. It follows that $[J^* : a] \supseteq I^*$ by the almost maximality of $I^*$. Suppose $\tau = (t_{ij}) \in [J^* : a]$, but $\tau = (t_{ij}) \in I^*$. Then the $p$-th row of $\sigma \tau$:

$$at_{p1}, at_{p2}, \ldots, at_{pn}, at_{p0}$$

are in $J$, but $t_{ps} \notin I$ for some $s, 1 \leq s \leq n$. Hence $[J : a] \supseteq I^*$.

(3) Now suppose $[I : a] \supseteq I$ and $b \in [I : a], b \notin I$. Let $\tau = (c_{ij}) \in I^*$, that is, $\tau = (c_{ij}) \in I^{**}$. Then

$$c_{p1}, c_{p2}, \ldots, c_{pn}$$

are in $I$, and

$$(aE_{pp})(\sum_j c_{ij}E_{ij}) = \sum_j aC_{pj}E_{pj}$$

is in $I^* = I^{**}$ since

$$aC_{p1}, aC_{p2}, \ldots, aC_{pn}$$

are in $I$. Hence $aE_{pp}I^* \subseteq I^*$. On the other hand,

$$aE_{pp}bE_{pp} = abE_{pp}$$

is in $I^* = I^{**}$ since $ab \in I$ by assumption. Hence

$$bE_{pp} \in [I^* : aE_{pp}],$$

since $b \notin I$. Therefore $aE_{pp} \in I^*$. We have now $a \in I$.

Now that we have established the existence of an almost maximal right ideal $I$ of $R$, it suffices to show that $I$ is faithful in order to prove the following theorem.

THEOREM 2. Let $R$ and $R_n$ be the same as in Theorem 1. Then the weak-primitivity of $R_n$ implies that of $R$.

Proof. Let $J^*$ be the faithful almost maximal right ideal of $R_n$ and let $M = R_n - J^*$ and $\alpha + J^* \in M$ for any $\alpha \in R_n$. Since $\gamma = b_{pp}E_{pp} \notin J^*$ where $\gamma$ is the same as in Lemma 3, there exist $\rho = (r_{ij}) \in R_n$ and $\sigma \in N(J^*)$ with $\sigma \notin J^*$, $\sigma + J^* \in N(J^*)/J^*$, such that

$$(b_{pp}E_{pp} + J^*)\rho = (\alpha + J^*)(\alpha + J^*)$$

by the weak transitivity in Lemma 1. Hence

$$\alpha + J^* = (\alpha + J^*)^{-1}(b_{pp}E_{pp} + J^*)\rho$$

$$= \delta(b_{pp}E_{pp} + J^*), \quad \delta = (\alpha + J^*)^{-1}.$$

Let $I$ and $I^*$ be the same as in Lemma 4. If $s \in [I : R]$, then $Rs \subseteq I$, and, in particular, $r_{pp}s \in I$. Hence $r_{pp}sE_{pp} \notin I^*$, since $I^* = I^{**}$. Now

$$(\alpha + J^*)(sE_{pp}) = [\delta(\gamma \rho + J^*)](sE_{pp})$$

or

$$\alpha sE_{pp} + J^* = \delta(b_{pp}r_{pp}sE_{pp} + J^*).$$

Now since

$$b_{pp}r_{pp}sE_{pp} = \gamma(r_{pp}sE_{pp})$$

we have $\delta = 0$. Therefore $r_{pp}sE_{pp} \in I$. Hence $s \notin I$. We have now $s \notin I$.

Now since

$$b_{pp}r_{pp}sE_{pp} = \gamma(r_{pp}sE_{pp})$$

we have $\delta = 0$. Therefore $r_{pp}sE_{pp} \in I$. Hence $s \notin I$. We have now $s \notin I$.
and \( \gamma(r_{pp}E_{pp}) \in J^* \), it follows that \( b_{pp}r_{pp}E_{pp} \in J^* \), i.e., \( b_{pp}r_{pp}E_{pp} + J^* = \overline{0} \). Hence

\[ \delta(b_{pp}r_{pp}E_{pp} + J^*) = \overline{0} \]

or

\[ \alpha sE_{pp} + J^* = \overline{0}. \]

Therefore \( \alpha sE_{pp} \in J^* \), which means \( sE_{pp} \in [J^* : R_n] \). But by the faithfulness of \( h \), \( [J^* : R_n] = (0) \). Hence \( s = 0 \) and consequently \( [I : R] = (0) \). This proves \( I \) is faithful.

5. Conclusion. Now we can conclude as follow:

**THEOREM 3.** Let \( R \) be a ring and \( R_n \) be the ring of \( n \times n \) matrices over \( R \), where \( n \) is a positive integer. Then \( R_n \) is weakly primitive if and only if \( R \) is weakly primitive.

**COROLLARY 1.** \( R_n \) is weakly transitive if and only if \( R \) is weakly transitive.

Proof. This follows from the fact that a ring is weakly primitive if and only if it is weakly transitive [3].

**COROLLARY 2.** If \( R_n \) (resp. \( R \)) is weakly primitive then \( R \) (resp. \( R_n \)) is prime.

Proof. This follows the fact that a weakly primitive is prime [3].

**References**


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