Openness Theorems on $C^r$ Immersions and Embeddings of Some Hilbert Manifolds

YONGTAE SHIN

1. Introduction. In this paper, we tacitly mean manifolds by Banach manifolds (without boundary) and denote them by $X$, $Y$, and etc. The set of all $C^r$ mappings of $X$ into $Y$ is denoted by $C^r(X, Y)$. An $f$ in $C^r(X, Y)$ is said to be a $C^r$ immersion if it is a locally $C^r$ diffeomorphism onto a submanifold $Z$ of $Y$, i.e. at each point $x$ in $X$ there exist open sets $U$ of $x$ in $X$ and $V$ of $f(x)$ in $Y$ and a submanifold $Z$ of $Y$ such that $f|_U : U \rightarrow Z \cap V$ is a $C^r$ diffeomorphism. If $f : X \rightarrow Z$ is a $C^r$ diffeomorphism (onto), then $f$ is called a $C^r$ embedding. We shall denote the set of all $C^r$ immersions and embeddings of $X$ into $Y$ by $\mathcal{G}(X, Y)$ and $\mathcal{E}(X, Y)$ respectively.

J. McAlpin's Embedding Theorem [5] says:

(1.2) Every second countable $C^r$ manifold modelled on a separable Hilbert space can be $C^r$ embedded onto a closed submanifold of a separable Hilbert space. (Cf. [3]).

It is a generalization of Whitney's Embedding Theorem [8]:

(1.2) There exists a $C^r$ embedding (resp. immersion) of a $C^r$ n-manifold onto a submanifold of a $p$-dimensional Euclidean space $R^p$ if $p \geq 2n+1$ (resp. $p \geq 2n$).

The author, by reducing (1.1), has proved a theorem [7]:

(1.3) If $X$ is a second countable $C^r$ manifold on an infinite dimensional separable Hilbert space $H$, then there exists a $C^r$ embedding of $X$ onto a closed submanifold of the Hilbert space $H' = H \times R$, the cartesian product of the $H$ and a real line $R$.

This theorem (1.3) could be considered as a generalization of the above (1.2) for some desirable form.

In a case of finite dimensional manifolds, L.S. Pontrijagin [6] proved a theorem:

(1.4) For a compact $C^r$ n-manifold $M$, $\mathcal{G}(M, R^m)$ (resp. $\mathcal{E}(M, R^{m+1})$) is open and dense in $\mathcal{G}(M, R^m)$ (resp. $\mathcal{E}(M, R^{m+1})$) with the topology induced by the $C^r$ sup. norm defined through the charts of $M$.

In a case of infinite dimensional manifolds, it is natural to ask:

Are the subsets $\mathcal{G}(X, H')$ and $\mathcal{E}(X, H')$ open and dense in $\mathcal{G}(X, H')$ with any topology induced naturally?

Received by the the editors, March 25, 1970.
The purpose of this paper is to prove the following theorems answering the part for openness of the above question.

**THEOREM 1.** Let \( X \) be a compact second countable \( C^r \) manifold, \( r \geq 1 \), modelled on an infinite dimensional separable Hilbert space \( H \). Then \( \mathcal{E}(X, H') \) is a non-empty open subset of the manifold \( \mathcal{E}'(X, H') \) with the natural atlas.

**THEOREM 2.** Under the same hypothesis, \( \mathcal{E}'(X, H') \) is a non-empty open subset of the manifold \( \mathcal{E}'(X, H') \) with the natural atlas.

2. The manifold structure of \( \mathcal{E}'(X, Y) \). For a manifold \( X \), we shall denote \( \tau_X: T(X) \to X \) and \( \tau^*_X: T^*(X) \to T(X) \) the (Banach) tangent bundles on \( X \) and \( T(X) \) respectively, and \( \Gamma(\pi) \) the set of all cross-sections of a (Banach) bundle \( \pi: P \to X \), where \( P \) is the (Banach) bundle space. A spray on \( X \) is a cross-section \( \xi \in \Gamma(\tau_2) \cap \Gamma(T^*_X) \) such that for all \( v \in T(X) \) and \( \lambda \in \mathbb{R} \), \( \xi_{\lambda v} = \lambda T h_\lambda(\xi(v)) \), where \( h_\lambda: T(X) \to T(X) \) defined by the correspondence \( v \mapsto \lambda v \). That is, a spray \( \xi \) on \( X \) is defined by the following commutative diagram:

\[
\begin{array}{ccc}
T^*(X) & \xrightarrow{\xi} & T^*(X) \\
\downarrow & & \downarrow \\
T(X) & \xrightarrow{h_\lambda} & T(X).
\end{array}
\]

For a spray \( \xi \) on \( X \) and a solution \( \alpha_r(t) \) of \( \xi \) at a point \( v \in T(X) \), with \( t \in (-a, a) \) an open interval, let \( \mathcal{D}_r \subset T(X) \) be the set of \( v \) in \( T(X) \) such that \( \alpha_r(1) \) is defined. The exponential of \( \xi \) is the mapping \( \exp^\xi: \mathcal{D}_r \to X \) defined by \( v \mapsto \tau_X(\alpha_r(1)) \) and we denote \( \exp^\xi(\tau_X, \exp^\xi): \mathcal{D}_r \times X \times X \) by \( v_x \mapsto (x, \exp^\xi(x)) \), where \( v_x \) is a point in the fibre over \( x \in X \). Therefore the mapping \( \exp^\xi \) is of class \( C^r \) if \( X \) is a \( \mathcal{E}' \) manifold, \( r \geq 1 \).

Now let \( X \) be a compact \( C^r \) manifold, \( r \geq 1 \), and \( Y \) a \( C^{r+2} \) manifold admitting partitions of unity. Let \( \xi \) be a \( C^{r+1} \) spray on \( Y \). Then there exist a neighborhood \( \mathcal{D}_\xi \subset T(X) \) of the zero cross-section on \( Y \) and a neighborhood \( \mathcal{F}_\xi \subset Y \times Y \) of the diagonal such that \( \exp^\xi|_{\mathcal{D}_\xi}: \mathcal{D}_\xi \to \mathcal{F}_\xi \) is a \( C^{r+2} \) diffeomorphism [1].

For each \( f \in \mathcal{E}'(X, Y) \), we can pull back the mapping \( \exp^\xi \) to the bundle space \( f^*T(Y) \) induced from \( T(Y) \) by \( f \). Thus we have a diffeomorphism \( \xi_f = f^*\exp^\xi: f^*\mathcal{D}_\xi \to \mathcal{D}_{f^*\xi} \), where \( \mathcal{D}_{f^*\xi} \subset X \times Y \) is a neighborhood of the graph \( (f) \). If we let \( U_{f, \xi} \) be the set of all \( g \in \mathcal{E}'(X, Y) \) such that graph \( (g) \subset \mathcal{F}_{f, \xi} \), the triple \( (U_{f, \xi}, \varphi_{f, \xi}, \Gamma'(f^*\tau_Y)) \) is a chart of \( \mathcal{E}'(X, Y) \) at \( f \), where \( \varphi_{f, \xi}: U_{f, \xi} \to \Gamma'(f^*\tau_Y) \) is defined by \( g \mapsto \xi_f^{-1}\text{graph}(g) \) and we know that the set \( \Gamma'(f^*\tau_Y) \) of all \( C^r \) cross-sections of the induced bundle \( f^*\tau_Y \) from the tangent bundle \( \tau_Y: T(Y) \to Y \) by \( f \) becomes a Banach space by the hypothesis that \( X \) is compact. The triple \( (U_{f, \xi}, \varphi_{f, \xi}, \Gamma'(f^*\tau_Y)) \)
is called a natural chart of $\mathcal{C}^r(X, Y)$ at $f$, and the maximal collection of all natural charts of $\mathcal{C}^r(X, Y)$ is called the natural atlas of $\mathcal{C}^r(X, Y)$. The following are a couple of the important results appeared in [1] as theorems:

1. If $X$ is a compact $C^r$ manifold, $r \geq 1$, and $Y$ a $C^{s-1}$ manifold admitting partitions of unity, the natural atlas of $\mathcal{C}^r(X, Y)$ is of class $C^s$, $0 \leq s \leq r$.

2. With the same hypothesis as above, the evaluation mapping

$$ev : \mathcal{C}^r(X, Y) \times X \to Y$$

defined by the correspondence $(f, x) \mapsto f(x)$ is of class $C^s$, $0 \leq s \leq r$.

3. Proof of Theorem 1. By the assumption given to the manifold $X$, $\mathcal{C}^r(X, H')$ is a non-empty subset of $\mathcal{C}^r(X, H')$, where $H' = H \times R$, since so is $\mathcal{E}^r(X, H')$ due to (1.3). On the other hand, since $H'$ is a separable Hilbert space that admits $C^r$ partitions of unity, $\mathcal{C}^r(X, H')$ is a $C^r$ manifold with the natural atlas due to (2.1).

We shall denote $L(E, F)$ the Banach space of all continuous linear mappings of a Banach space $E$ into a Banach space $F$ and $IL(E, F)$ the subset of $L(E, F)$ consisting of all splitting injections (i.e., injective mappings whose images split in $F$). The following is proved in [1].

**Lemma 1.** $IL(E, F)$ is a non-empty open subset of the Banach space $L(E, F)$.

Due to (2.2) the evaluation mapping $ev : \mathcal{C}^r(X, H') \times X \to H'$ is of class $C^s$, and if we consider a partial mapping with respect to the second factor $ev : \mathcal{C}^r(X, H') \times X \to H'$, it is equivalent to the mapping $f : X \to H'$ for each $f \in \mathcal{C}^r(X, H')$. Thus $D_{x} ev : \mathcal{C}^r(X, H') \times X \to L(H, H')$ is a $C^{s-1}$ mapping such that $D_{x} ev(f, x) = T_{x} f$, where $H$ is identified with $T_{x}(X)$ for all $x \in X$ and $D_{x}$ is the derivative operator with respect to the second factor. By Lemma 1, $\mathcal{C} = (D_{x} ev)^{-1}(IL(H, H'))$ is open in $\mathcal{C}^r(X, H') \times X$. Suppose $f$ is in $\mathcal{C}^r(X, H')$ it is obvious that $\{f\} \times X \subset \mathcal{C}$ by the usual criterion for immersions known as the Implicit Function Theorem on manifolds.

Since $\mathcal{C}$ is open, for each $x \in X$ we have an open subset $\mathcal{V}^{-1}(f) \times U(f)$ of $\mathcal{C}^r(X, H') \times X$ such that $\mathcal{V}^{-1}(f) \times U(x)$ is contained in $\mathcal{C}$ with $\mathcal{V}^{-1}(f)$ and $U(x)$ being open neighborhoods of $f$ and $x$ respectively. Let $\{U(x_i)\}_{i=1}^{n}$ be a finite open covering of $X$ and $\mathcal{V}^{-1}(f)$ the corresponding open sets in $\mathcal{C}^r(X, H')$. If we let $\mathcal{V}(f) = \bigcap_{i=1}^{n} \mathcal{V}^{-1}(f)$, $\mathcal{V}(f)$ is an open neighborhood of $f$ in $\mathcal{C}^r(X, H')$ and moreover, for each $g \in \mathcal{V}(f)$ we have $\{g\} \times U(x_i) \subset \mathcal{C}$ for each $i$, since $\mathcal{V}^{-1}(f) \times U(x_i) \subset \mathcal{C}$ for each $i$. Taking the union for all $i$, $1 \leq i \leq n$,

$$\bigcup_{i=1}^{n} \{g\} \times U(x_i) = \{g\} \times \bigcup_{i=1}^{n} U(x_i) = \{g\} \times X \subset \mathcal{C}.$$

Therefore $\mathcal{V}(f) \times X \subset \mathcal{C}$. This implies that $\mathcal{V}(f) \subset \mathcal{C}^r(X, H')$, and hence
\( \mathcal{V}(X, H') \) is open in \( \mathcal{C}^r(X, H') \).

4. Proof of Theorem 2. If we denote \( BL(E, F) \) the subset of the \( L(E, F) \) consisting of all linear isomorphisms, where \( E \) and \( F \) are any Banach spaces as before, the following result is known \([4]\):

**Lemma 2.** \( BL(E, F) \) is a non-empty open subset of \( L(E, F) \).

By its definition if \( f \in \mathcal{C}^r(X, H') \), then there exists a submanifold \( Z \) of \( H' \) such that \( f : X \rightarrow Z \) is a \( \mathcal{C}^r \) diffeomorphism of \( X \) onto \( Z \), and moreover \( T_x f : H_x \rightarrow H' \) is a linear isomorphism (into). Thus considering the evaluation mapping \( ev: \mathcal{C}^r(X, H') \times X \rightarrow H' \) and its partial derivative \( D_x ev \) with respect to the second factor as before, it is not difficult to prove that \( \mathcal{C}^r(X, H') \) is open in \( \mathcal{C}^r(X, H') \) by the similar way done for the proof of Theorem 1 due to Lemma 2.

**References**


Choongnam University