A Measurement of Political Power in Voting

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Introduction. In a committee which is a decision making body, the power of individual members can be computed by considering various alignments of the committee members on a number of issues. The k members are ordered $m_1, m_2, \ldots, m_k$ according to how likely they are to vote for the measure. If the measure is to carry, we must persuade m_1 and m_2 up to m_i to vote for it until we have a winning coalition. If the set $m_1, m_2, \ldots, m_i$ is a winning coalition but $m_1, m_2, \ldots, m_{i-1}$ is not, then m_i is the crucial member of the coalition and we call m_i the *pivot*. Therefore, it is reasonable to define the (voting) power of a member of a committee to be the number of alignments in which he is pivotal, divided by the total number of alignments ($k!$ for a committee of k members).

Since in each alignment some one must be pivotal and the total number of alignments is $k!$, the sum total of the powers of members must be $k!/k!=1$.

Suppose $k=3$, i.e., there are three members.

(1) If each has one vote and a measure is passed by majority, then the second position in the sequence of m_1, m_2, m_3 is pivotal and there are $2!$ ways each member to be pivotal (*e.g.*, m_1, m_2, m_3 or m_3, m_2, m_1 are only two occasions in which m_2 becomes pivotal). Hence the power of m_2, as with the other members, is $2!/3!=\frac{1}{3}$.

(2) Suppose m_1 has three votes while m_2 and m_3 have a single vote each. Then m_1 is always pivotal regardless his position, for every alignment. Thus he has the absolute power 1. In such a case m_1 is called a dictator.

(3) Suppose m_1 has two votes and the other members have single votes.

Then either m_2 or m_3 can be pivotal provided m_1 precede both. Since there is only one sequence for which m_2 (or m_3) becomes pivotal, m_2 has the power $\frac{1}{3!}=\frac{1}{6}$. The sequences for which m_1 is pivotal are m_2, m_1, m_3; m_3, m_1, m_2; m_2, m_3, m_1; m_3, m_2, m_1. Hence the power of m_1 is $\frac{4}{3!}=\frac{2}{3}$, and consequently the power of each of m_2 and m_3 shares $\frac{1}{6}$. Thus their ratio is 4 : 1 : 1. Note that the number of votes one possesses is not necessarily directly proportional to his power index.

Power Structure I. Part 1: The Security Council of the United Nations previously consisted of the Big Five veto powers and six small nation members. In order that a measure be passed by the Council, seven members including all of he Big Five must vote for the measure.

First, we shall find the power of a small nation member by computing the number of alignments in which she becomes pivotal. Denote the Big Five and six small nations by B_1, B_2, B_3, B_4, B_5; $s_1, s_2, s_3, s_4, s_5, s_6$. Now, suppose s_1 is pivotal. Then five B_j's and one s_i must precede s_1 and the remaining four s_i's must follow. After one s_j has been selected (there are 5 choices), there are 6! ways to arrange (permute) the preceding members and 4! ways to arrange the following four members. Hence there are exactly 5(6!)(4!) alignments in which s_1 becomes the pivot. Thus its power index, the same for all small nation members, is:

$$5(6!)(4!)/(11!)=1/462=0.00216$$

Also, since clearly the total power is 1 and the sum total power of six small nations is 6/462=1/77, each of the Big Five shares,

$$(1-1/77)/5=76/385=0.19740$$

The above power indices show that nearly all the power is in the hands of the Big Five.

Part 2: Several years ago the Security Council went through its own structural change. And now it consists of the same Big Five, but with ten small nations, instead of six, and nine votes are needed to carry the measure (including, of course, those Big Five).

Computing again as in part 1) we obtain as the power of a small nation:

$$\binom{9}{3}8!6!/15!=4/2145\doteq 0.00186$$

and as the power of Big one

$$(1-40/2145)/5=2105/10725\doteq 0.19627$$

Thus each of the Big Five suffered a slight loss in power (by an amount of 0.00113) by the change.

Power Structure II. Part 1. We consider this time an institute, similar to a college, in which five Division Chairmen, the Dean and the President form a policy-making (executive) committee. Suppose that both the Dean and the President have veto powers and that five yes-votes, including those of the two veto members, are required to pass a measure. Then the power index of each Division Chairman is: $\binom{4}{2}$ $4!2!/7!=2/35\doteq 0.0571$
and that of either the Dean or the President is:

$$(1-10/35)/2=5/14\doteq 0.3571.$$

In case a Division Chairman's vote is determined by his own divisional meeting, prior to the meeting of the above mentioned committee, which is attended by several Heads of Departments, then the Chairman's power shall be weakened accordingly. For instance, if the divisional meeting, consisting of six Department Heads and its Chairman as the sole veto member, passes an issue by majority, then each Department will share $\binom{6}{2}3!3!/7!=3/28=10.7\%$ of the above Division power, which is $\frac{2}{35}$, i.e., each Head's voting power is $(3/28)(2/35)\doteq 0.00612$ (less than 1%). In this case the Division Chairman's power is reduced to

$$(1-18/28)(2/35)=1/49\doteq 0.0204 \text{ from } 0.0571.$$

Even though the power of a Department Head is insignificant with respect to the whole power structure of the institute, he can have an absolute power over professors who belong to his department, on certain matters. The relative power of a professor to that of his Department Head decreases obviously as his influence on departmental affairs diminishes.

Part 2: Now we investigate a structural change caused by creation of a Senate (to initiate constructive dialogues and thus to promote democracy) within the institute. Assume that the Senate is composed of eleven elected Professors, outside of the above mentioned executive officials, and most importantly assume that the President and the Senate alone can carry a measure for which both agree, independently from the executive committee discussed above. Let us call the committee consisting of the President and the Senate the *Presenate* committee. (It is considered to consist of two members.)

Since the two committees are itndependent and exclusive of one another in their functions, each one shares precisely one half of the total power 1. Since the Senate clearly shares one half of the power that the Presenate committee can exert, the Senate enjoys $\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$ of the total power, and thus each senator enjoys $1/44\doteq0.0227$. After this change a Division Chairman's power, without Divisional meeting, shall be decreased by half to $1/35\doteq0.0285$ (a bit greater than a senator) and that of the Dean to $5/28\doteq0,1786$, whereas that of the president shall be increased to 0.4286 from 0.3571.

Part 3: In case the Dean joins the presenate committee to form P-D-S (three member-) committee in which only the President has the veto power, the distribution of power for the President, the Dean, a Division Chairman and a senater will be 0.5118, 0.2618, 0.0285 and 0.0075, respectively. By this change, we note that the powers of the President and the Dean increase, whereas that of a senator falls from the level of a Division Chairman to that of a Department Head.

Finally, we note that the power indices which appeared in Power Structure II are computed under either ideal or simplified situations based on several assumptions (*e.g.*, in its decision making rules). Apparently, the President's power is immensely greater if we consider numerous measures which need not filter through any of the above mentioned channels (assuming this institute is completely autonomous).

We assumed also, with a slight mystifying effect, that the two committees share equal power because simply they have equal potential. In reality, however, the relative power of one committee depends on its relative amount of activity, relative to the other committee.

References

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