A CHARACTERIZATION OF THE BEHRENS RADICAL

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1. Introduction

In 1954 E.A. Behrens [3] introduced a radical class $B$ lying properly between the Jacobson radical class $J$ and the Brown-McCoy radical class $G$. Behrens [3] defined the $B$-radical, $B(R)$, of a ring $R$ as follows: $B(R) = \{y \in R : y \subseteq (y^2 - y) \text{ for all } y \subseteq (x)\}$, where $(y^2 - y)$ and $(x)$ denote the principal ideals of $R$ generated by the elements $y^2 - y$ and $x$, respectively. N.J. Dwinsky in [4] presented $B$ as the upper radical class $\Phi(M)$ determined by the special class $M = \text{all subdirectly irreducible rings with idempotent hearts such that the hearts contain non-zero idempotent elements}$. In this paper we give a somewhat simpler characterization of the Behrens radical class.

We shall employ the following notation throughout.

- $H(R)$ denotes all homomorphic images of the ring $R$.
- $U(R)$ denotes the heart of the ring $R$.
- $I \subseteq R$ denotes $I$ is an ideal of the ring $R$.
- $I \unlhd R$ denotes $I \subseteq R$ but $I \cong R$.
- $R \cong R'$ denotes the rings $R$ and $R'$ are isomorphic.
- $0$, depending upon the context in which it appears, denotes the ring $0$, the ideal $0$, or the class $\{0\}$.

We shall use the following characterization of radical classes [1, p. 105]. A subclass $P$ of a universal class $W$ of rings is a radical class if and only if $P$ satisfies the following three conditions.

(i) $P$ is homomorphically closed.
(ii) If $\{I_\alpha : \alpha \in \Gamma\}$ is a chain of $P$-ideals of a ring $R \in W$, then $\bigcup_{\alpha \in \Gamma} I_\alpha$ is a $P$-ideal of $R$.
(iii) If $R \in W$ and if $I \subseteq R$ such that $I \in P$ and $R/I \in P$, then $R \in P$.

Let $W$ be a universal class of rings and define a subclass $B^*$ of $W$ by $B^* = \{R \in W : R$ has no homomorphic image with non-zero idempotent elements\}.

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2. Theorems

THEOREM 1. The class $B^*$ is a radical class.

PROOF. We shall show that each of the conditions (i), (ii), (iii) is satisfied by $B^*$. First let $R \in H(B^*)$ and let $g(R)$ be a non-zero homomorphic image of $R$. If $g(R)$ contained a non-zero idempotent element, then so would the homomorphic image of some ring in $B^*$. Thus $R$ must be in $B^*$ and hence $B^* = H(B^*)$.

To show that $B^*$ satisfies condition (ii), let $\bigcup I_\alpha \subseteq B^*$. Then there exists a homomorphism $f$ and an element $x \in \bigcup I_\alpha$ such that $f(x) = e = e^2 = 0$. Now $x \in I_\alpha$ for some $\alpha$, and $e = f(x) \in f(I_\alpha) \subseteq H(B^*) = B^*$. Thus $f(I_\alpha)$ is a non-zero homomorphic image of $I_\alpha \subseteq B^*$ and $f(I_\alpha)$ has a non-zero idempotent element—a contradiction to the assumption that $I_\alpha \subseteq B^*$. Hence $\bigcup I_\alpha \subseteq B^*$ and condition (ii) is satisfied.

Finally let $R \in W$, $I \subseteq R$, $I \subseteq B^*$, $R/I \in B^*$. We must show that $R \in B^*$. Again, by way of contradiction, assume that $R \not\in B^*$. Then let $f$ be a homomorphism and let $0 \neq e^2 = e \in f(R)$. Now $e \in f(I)$, because $I \subseteq B^*$. So $f(R)/f(I)$ contains the non-zero idempotent element $e + f(I)$. But then $R/I$ may be mapped homomorphically onto the ring $f(R)/f(I)$ by the homomorphism $\hat{f}$ defined by $\hat{f}(x + I) = f(x) + f(I)$, while $R/I \in B^*$. This contradiction forces $R$ to be a member of $B^*$.

THEOREM 2. The radical class $B^*$ is hereditary.

PROOF. Let $R \in B^*$ and let $0 \neq I \subseteq R$. If $I \subseteq B^*$, then I has a homomorphic image $I/K$ with a non-zero idempotent element $x + K$, i.e., $K \triangleleft I$, $x \in I$, $x \in K$, $x^2 - x \in K$. Now $x^2 \in K$; for if $x^2 \notin K$, then $-x \notin K$ and hence $x \notin K$. Let $K'$ denote the ideal of $R$ generated by $K$. By [2, p. 186] we have $(K')^3 \subseteq I$. Thus if $x \in K'$, then $x^3 \in (K')^3 \subseteq K$. Since $x \in I$ and $x^2 - x \in K \subseteq I$, we have $x^3 - x^2 \in xK \subseteq K$. But then $x^2 \in K$. Thus $x \in K'$. But $x^2 - x \in K \subseteq K'$ and $x \in K'$ imply that $R/K'$ has a non-zero idempotent element $x + K'$. This is contrary to the assumption that $R \in B^*$. Hence we must have $I \in B^*$ and thus that $B^*$ is hereditary.

LEMMA 1. [4, Lemma 74]. Let $P$ be a hereditary radical. Then a subdirectly irreducible ring $R$ with heart $\mathcal{H}(R)$ is $P$-semi-simple if and only if $\mathcal{H}(R)$ is $P$-semi-simple.
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LEMMA 2. $B^* \subseteq B$.

PROOF. Since every ring in $M$ is $B$-semi-simple and since $B$ is the upper radical determined by the class $M$ it suffices to show that every ring in $M$ is $B^*$-semi-simple. For this let $R \in M$. Then $0 \neq \mathcal{Z}(R)$ and $\mathcal{Z}(R)$ contains a non-zero idempotent element. Thus $B^*(\mathcal{Z}(R)) = 0$, and so by Lemma $B^*(R) = 0$.

NOTE. Let $E$ be the class of all rings whose hearts have non-zero idempotent elements. Then $B^* \subseteq \mathcal{E}(E)$, the upper radical determined by the class $E$. Clearly, since $M \subseteq E$ we have $\mathcal{E}(E) \subseteq \mathcal{E}(M) = B$.


PROOF. Let $R \in B$ and assume that $R \not\subseteq B^*(R)$. Now $R \subseteq B^*$ implies that $R/B^*(R)$ is not zero. Since $R/B^*(R)$ is $B^*$-semi-simple, $R/B^*(R)$ has a non-zero homomorphic image $R/I$ with a non-zero idempotent element $x + I(B^*(R) \subseteq I$, $x^2 - x \in I$, $x \in I$). But $R \in B$, so we must have $x \in (x^2 - x) \subseteq I$. This is a contradiction. Hence $R \in B$ implies $R \in B^*$, i.e., $B \subseteq B^*$. By Lemma 2 $B^* = B$.

THEOREM 4. Let $R$ be a ring. Then $B^*(R) = \bigcup \{I \subseteq R \mid I \subseteq R$ and every non-zero ideal of $R/I$ can be mapped homomorphically onto a ring with a non-zero idempotent element}.

PROOF. Since $R/B^*(R)$ is $B^*$-semi-simple, each non-zero ideal of $R/B^*(R)$ is $B^*$-semi-simple. Therefore each non-zero ideal of $R/B^*(R)$ can be mapped homomorphically onto a ring with a non-zero idempotent element. Now let $I$ be any ideal of $R$ such that each non-zero ideal of $R/I$ can be mapped homomorphically onto a ring with a non-zero idempotent element. We show that $B^*(R) \subseteq I$. By way of contradiction assume that $B^*(R) \not\subseteq I$. Then $0 \neq (B^*(R) + I)/I \subseteq R/I$ and hence $(B^*(R) + I)/I$ can be mapped homomorphically onto a ring with a non-zero idempotent element. But $(B^*(R) + I)/I \approx B^*(R)/B^*(R) \cap I \subseteq B^*$, i.e., $B^*(R)/B^*(R) \cap I$ can be mapped homomorphically onto a ring with a non-zero idempotent element. This is a contradiction, because $B^*(R)/B^*(R) \cap I \subseteq B^*$. Hence $B^*(R) \subseteq I$ and so $B^*(R) \subseteq I \subseteq R$ and every non-zero ideal of $R/I$ can be mapped homomorphically onto a ring with a non-zero idempotent element]. But $B^*(R)$ is such an ideal $I$ and so equality obtains.
REFERENCES


