A TOPOLOGICAL PROPERTY OF THE REALS

By Norman Levine

A fundamental property of the reals with the standard topology is that the rationals and the irrationals are both dense in the reals. It is this property which motivates the author to make the following.

DEFINITION. A topological space $(X, \mathscr{T})$ has property R (or is an R-space) iff there exists a set $A \subset X$ such that $c(A)=X=c(\mathscr{C}A)$, c denoting the closure operator and $\mathscr{C}$ denoting the complement operator.

In this paper we give several characterizations of R-spaces and investigate some of their stable properties.

THEOREM 1. *A space $(X, \mathscr{T})$ is an R-space iff there exists a set $A \subset X$ such that Int $A=\phi=$ Int $\mathscr{C}A$, Int denoting the interior operator.*

PROOF. $c(A)=X=c(\mathscr{C}A)$ iff $\mathscr{C}c(A)=\phi=\mathscr{C}c(\mathscr{C}A)$ iff $\mathscr{C}c(\mathscr{C}\mathscr{C}A)=\phi=\mathscr{C}c(\mathscr{C}A)$ iff Int $\mathscr{C}A=\phi=$Int A.

From theorem 1, we get easily

COROLLARY 2. *If $(X, \mathscr{T})$ has property R, then $\{x\} \in \mathscr{T}$ for no $x \in X$.*

The converse of corollary 2 is false as is seen in

EXAMPLE 3. Let X be an infinite set and suppose that $\mathscr{F}$ is the cofinite filter on X. It is easy to see that $\bigcap \mathscr{F}=\phi$. Now $\mathscr{F}$ is contained in an ultrafilter $\mathscr{F}^*$; let $\mathscr{T}=\{\phi\} \cup \mathscr{F}^*$. $\mathscr{T}$ is clearly a topology for X. For each set $A \subset X$, either $A \in \mathscr{F}^*$ *or* $\mathscr{C}A \in \mathscr{F}^*$ and hence Int $A \neq \phi$ or Int $\mathscr{C}A \neq \phi$. Thus, by theorem 1, $(X, \mathscr{T})$ is not an R-space. For each $x \in X$, $\{x\} \in \mathscr{T}$ holds for no x; for if $\{y\} \in \mathscr{T}$, then $\{y\} \in \mathscr{F}^*$ and $y \in \bigcap \mathscr{F}^* \subset \bigcap \mathscr{F}=\phi$, a contradiction.

In example 3, it was essential that we take an infinite set X, for consider

THEOREM 4. *Let $(X, \mathscr{T})$ be a finite topological space such that $\{x\} \in \mathscr{T}$ holds for no $x \in X$. Then $(X, \mathscr{T})$ is an R-space.*

PROOF. This follows immediately from theorem 5.

THEOREM 5. *Let $(X, \mathscr{T})$ be a topological space and suppose that $\mathscr{T}$ is finite.*

If $\{x\} \in \mathscr{T}$ for no $x \in X$, then $(X, \mathscr{T})$ is an R-space.

PROOF. This follows immediately from theorem 6.

THEOREM 6. *Let $(X, \mathscr{T})$ be a topological space with the property that $\{x\} \in \mathscr{T}$ for no $x \in X$. Suppose further that every non empty open set contains a minimal non empty open set. Then $(X, \mathscr{T})$ is an R-space.*

PROOF. Consider the collection $\mathfrak{A}$ of all minimal non empty open sets O^*. It is clear that $\mathfrak{A}$ is a pairwise disjoint collection of sets and that every O^* in $\mathfrak{A}$ contains at least two points. For each O^* in $\mathfrak{A}$, select a point $x^* \in O^*$ and let A be the set of points thus chosen. It is clear that $\text{Int}\, A = \phi = \text{Int}\, \mathscr{C}A$ and by theorem 1, $(X, \mathscr{T})$ is an R-space.

In theorem 7, it will be convenient to have the following

DEFINITION. In a space $(X, \mathscr{T})$, we let $\mathscr{P} = \{(A, B) : A \neq \phi \neq B,\ A \cap B = \phi,\ A \subset c(B),\ B \subset c(A)\}$. We partially order $\mathscr{P}$ as follows: $(A, B) \leqq (C, D)$ iff $A \subset C$ and $B \subset D$.

THEOREM 7. *A space $(X, \mathscr{T})$ is an R-space iff for each non empty open set O in X, there exists a pair $(A, B) \in \mathscr{P}$ such that $A \cup B \subset O$.*

PROOF. Suppose that $(X, \mathscr{T})$ is an R-space and let O be a non empty open set. There exists a set E such that $c(E) = X = c(\mathscr{C}E)$. Let $A = O \cap E$ and let $B = O \cap \mathscr{C}E$. Now $c(B) = c(O \cap \mathscr{C}E) = c(O) \supset O \supset A$. Likewise $B \subset c(A)$. It follows that $(A, B) \in \mathscr{P}$ and that $A \cup B \subset O$.

Conversely suppose that for each non empty open set, there exists a pair $(A, B) \in \mathscr{P}$ such that $A \cup B \subset O$. It follows then that $\mathscr{P} \neq \phi$. We show now that $\mathscr{P}$ has a maximal element. To this end, let $\mathscr{B}$ be a non empty simply ordered subset of $\mathscr{P}$. Let $A^* = \bigcup\{A : (A, B) \in \mathscr{B}$ *for some set* $B\}$ and let $B^* = \bigcup\{B : (A, B) \in \mathscr{B}$ *for some set* $A\}$. Clearly, $A^* \neq \phi \neq B^*$, $A^* \cap B^* = \phi$ and $c(A^*) \supset c(A) \supset B$ *for all* $(A, B) \in \mathscr{B}$ and hence $c(A^*) \supset B^*$. Similarly, $c(B^*) \supset A^*$. Hence $(A^*, B^*) \in \mathscr{P}$ and (A^*, B^*) is an upper bound for $\mathscr{B}$. Let $(A^\#, B^\#)$ be maximal in $\mathscr{P}$. To show that $(X, \mathscr{T})$ is an R-space, it suffices to show that $c(A^\#) = X = c(B^\#)$ for then $c(\mathscr{C}A^\#) = X$. Suppose then that $c(A^\#) \neq X$. Then $O = X - c(A^\#)$ is a non empty open set and by assumption, there exists a pair $(E, F) \in \mathscr{P}$ such that $E \cup F \subset O$. Since $B^\# \subset c(A^\#)$, it follows that $(A^{*} \cup E, B^{*} \cup F) \in \mathscr{F}$ and (A^{*}, B^{*}) is not maximal, a contradiction.

COROLLARY 8. *Let $(X, \mathscr{T})$ be locally arcwise connected. Then $(X, \mathscr{T})$ has property R.*

COROLLARY 9. *$(X, \mathscr{T})$ is an R-space if $X = \bigcup\{A_\alpha : \alpha \in \Delta\}$, where $(A_\alpha, A_\alpha \cap \mathscr{T})$ is an R-space for each $\alpha \in \Delta$.*

PROOF. Let O be a non empty open subset of X. Then $O \cap A_\alpha \neq \phi$ for some $\alpha \in \Delta$. Hence there exist sets $A \neq \phi \neq B$ such that $A \cap B = \phi$, $A \cup B \subset O \cap A_\alpha$ and $c_\alpha(A) \supset B$ and $c_\alpha(B) \supset A$. It follows then that $A \cup B \subset O$ and that $c(A) \supset B$, $c(B) \supset A$.

COROLLARY 10. *Let $(X, \mathscr{T})$ be an R-space and suppose that O is a non empty open set. Then $(O, O \cap \mathscr{T})$ is an R-space.*

We omit the easy proof.

COROLLARY 11. *Let $X = \bigcup\{O_\alpha : \alpha \in \Delta\}$, where O_α is a non empty open set for each $\alpha \in \Delta$. Then X is an R-space iff O_α is an R-space for each $\alpha \in \Delta$.*

COROLLARY 12. *Let Y be an R-space and suppose that Y is dense in X. Then X is an R-space.*

PROOF. Let O be a non empty open subset of X. Then $O \cap Y$ is a non empty open subset of Y and hence there exist sets $A \neq \phi \neq B$ such that $A \cap B = \phi$, $A \cup B \subset O \cap Y$ and $A \subset c_Y(B)$ and $B \subset c_Y(A)$. It follows then that $A \cup B \subset O$ and $A \subset c(B)$, $B \subset c(A)$.

THEOREM 13. *Let $f : X \to Y$ be an open transformation (continuity not assumed). If Y is an R-space, then X is an R-space.*

PROOF. There exists a set A such that $Y = A \cup \mathscr{C}A$ and $\operatorname{Int} A = \phi = \operatorname{Int} \mathscr{C}A$ by theorem 1. Then $X = f^{-1}[A] \cup f^{-1}[\mathscr{C}A]$, $f^{-1}[A] \cap f^{-1}[\mathscr{C}A] = \phi$ and $\operatorname{Int} f^{-1}[A] = \phi = \operatorname{Int} f^{-1}[\mathscr{C}A]$. Applying theorem 1 again, X is an R-space.

COROLLARY 14. *Let $(X, \mathscr{T}) = \times\{(X_\alpha, \mathscr{T}_\alpha) : \alpha \in \Delta\}$. If $(X_\alpha, \mathscr{T}_\alpha)$ is an R-space for at least one $\alpha \in \Delta$, then $(X, \mathscr{T})$ is an R-space.*

The converse of corollary 14 is false as seen in

EXAMPLE 15. Let Δ be an infinite set and for each $\alpha \in \Delta$, let $X_\alpha = \{a, b\}$ and let $\mathcal{T}_\alpha = \{\phi, \{a\}, X_\alpha\}$. If $(X, \mathcal{T}) = \times\{(X_\alpha, \mathcal{T}_\alpha) : \alpha \in \Delta\}$, then $(X, \mathcal{T})$ is an R-space although $(X_\alpha, \mathcal{T}_\alpha)$ is an R-space for no $\alpha \in \Delta$. To see that $(X, \mathcal{T})$ is an R-space, let $A = \bigcap\{P_\alpha^{-1}[a] : \alpha \in \Delta\}$. Then Int $A = \phi =$ Int $\mathscr{C}A$.

We conclude with two sufficient conditions for a space to be an R-space.

THEOREM 16. *$(X, \mathcal{T})$ is an R-space if it is separable and every non empty open set is uncountable.*

THEOREM 17. *$(X, \mathcal{T})$ is an R-space if there exists a countable base $\{O_i : i \in P\}$ with the property that* card $O_i \geqq 2i$ *for each i.*

PROOF. Let $x_1 \neq y_1$ in O_1. Pick x_2, y_2 in O_2 such that card $\{x_1, x_2, y_1, y_2\} = 4$. In general, we can choose x_i, y_i in O_i so that card $\{x_1, \cdots, x_i, y_i, \cdots, y_i\} = 2i$. If $A = \{x_i : i \in P\}$, then $c(A) = X = c(\mathscr{C}A)$.

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