L-REGULAR SEMIGROUPS, I

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Let E be an idempotent semigroup (a band). The collection $E(R)$ of $\mathscr{R}$-classes of E may be partially ordered by the following rule: if $E_1, E_2 \in E(R), E_1 < E_2$ iff $e < f$ for all $e \in E_1$ and $f \in E_2$ ($e \leqq f$ iff $ef = fe = e$). E is termed *lexicographically ordered* if $E(R) = \{E_{(n,\alpha)} : n \in I^0$, the non-negative integers, $\alpha \in Y$, a semilattice with greatest element$\}$ with $E_{(n,\alpha)} < E_{(m,\beta)}$ iff $n > m$ or $n = m$ and $\alpha < \beta$ and $E_{(n,\alpha)} E_{(n,\beta)} \subset E_{(n,\alpha\wedge\beta)}$ for α and β non-comparable in Y. A regular semigroup is termed *L-regular* if its idempotents form a lexicographically ordered band and $E_{(n,\alpha)}$ and $E_{(m,\beta)}$ are contained in the same $\mathscr{D}$-class of S iff $\alpha = \beta$. We determine the structure of simple L-regular semigroups mod L-semilattices of right groups (theorem 1). From theorem 1, we deduce the structure of simple E-regular semigroups mod E-chains of right groups (corollary 1; see also [4]) and the structure of simple L-inverse semigroups mod L-semilattices of groups (corollary 2; cf [2, theorem 3.3]).

Unless otherwise specified, we use the notation and terminology of [1]. If X is a semigroup, E_X will denote the set of idempotents of X.

We first make some introductory remarks. Let S be a simple L-regular semigroup. We may label the $\mathscr{D}$-classes of S as $\{D_\alpha : \alpha \in Y\}$ with $E_{D_\alpha} = \{E_{(n,\alpha)} : n \in I^0\}$. From each $E_{(n,\alpha)}$ choose an element $e_{(n,\alpha)}$. The $\mathscr{R}$-classes of S contained in D_α may be labeled $\{Re_{(n,\alpha)} : n \in I^0\}$ and the $\mathscr{L}$-classes of S contained in D_α may be labeled $\{L_f : f \in E_{(n,\alpha)},\ n \in I^0\}$. It is easily seen that each $\mathscr{R}$-class ($\mathscr{L}$-class) of S contained in D_α is an $\mathscr{R}$-class ($\mathscr{L}$-class) of D and conversely. By [5, lemma 1.1], each D_α is an E-bisimple semigroup (If E (R) is order isomorphic to I under the reverse of the usual order, E is called *naturally ordered band*. A bisimple semigroup S such that E_S is a naturally ordered band is termed an E-bisimple semigroup.). Hence, the structure of each D_α is known mod groups [3, theorem 1.2]. Let us define $t_{((n,\alpha),(m,\alpha))} = \cup \{R_{e(n,\alpha)} \cap L_f : f \in E_{(m,\alpha)}\}$. If $a \in R_e \cap L_f$ where $e, f \in E_S$, there exists a unique inverse a^{-1} of a contained in $R_f \cap L_e$ such that $aa^{-1} = e$ and $a^{-1}a = f$ [1, theorem 2.18]. Let α_0 denote the greatest element of Y.

In [4], we considered the special case where Y was the finite chain $0 > 1 > 2 > \cdots$

$>d-1$ where d is a positive integer. Surprisingly, in most cases the lemmas and their proofs in [4] may be extended to the more general situation with only small modifications. In particular, in [4] make replacements "E"$_{(n,\alpha)}$ for "E_{i+nd}", "e"$_{(n,\alpha)}$ for "e"$_{i+nd}$, "α_0" for "0", etc. The proofs of lemmas that may be obtained from corresponding lemmas of [4] in this manner will be omitted. The main structure theorem (theorem 1) will be a consequence of 12 lemmas.

LEMMA 1. (cf., [4, lemma 2]). *Let* $a\in R_{e(0,\alpha_0)}\cap L_{e(1,\alpha_0)}$. *Then,* $e_{(0,\alpha)}\,a\in t_{((0,\alpha),\ (1,\alpha))}$ *and* $a^{-1}\ e_{(0,\alpha)}\in t_{((1,\alpha),(0,\alpha))}$.

REMARK 1, Let $a_\alpha=e_{(0,\alpha)}a$. Then, as in [4, remark2], $a_\alpha^{-1}=a^{-1}e_{(0,\alpha)}$, $a_\alpha^n=e_{(0,\alpha)}a^n$, and $a_\alpha^{-n}=a^{-n}\ e_{(0,\alpha)}$ for $n\in I^0$. This comment is used in establishing several of the lemmas.

LEMMA 2. (cf., [4, lemma 3]) $t_{((k,\alpha),\ (n,\alpha))}\ t_{((r,\alpha),\ (s,\alpha))}\subset t_{((k+r-min(n,r),\alpha),\ (n+s-min(n,r),\alpha))}$.

LEMMA 3. (cf. [4, lemmas 4-6]) *If* $g\in t_{((k,\alpha),(k,\alpha))}$, *then* $ga^{-r}\in t_{((r,\alpha_0),\ (0,\alpha_0)}$ *if* $r>k$ *and* $ga^{-k}\in t_{((k,\alpha),\ (0,\alpha))}$.

LEMMA 4. (cf., [4, lemma 7]). *If* $g\in t_{((k,\alpha),(k,\alpha))}$, $ga^s\in t_{((k,\alpha),\ (k+s,\alpha))}$.

LEMMA 5. (cf., [4, lemma 8]). *Every element of S may be uniquely expressed in the form* $a^{-n}a^k\ g_{k\alpha}$ *where* $g_{k\alpha}\in t_{((k,\alpha),\ (k,\alpha))}$ *and* $a^{-n}a^k g_{k\alpha}\in t_{((n,\alpha),\ (k,\alpha))}$.

LEMMA 6.

$$t_{((n,\alpha),(n,\alpha))}\ t_{((m,\beta),\ (m,\beta))}\subset\begin{cases}t_{((n,\alpha),(n,\alpha))} & \text{if } n>m\\ t_{((m,\beta),(m,\beta))} & \text{if } m>n\\ t_{((n,\alpha\wedge\beta),(n,\alpha\wedge\beta))} & \text{if } m=n.\end{cases}$$

PROOF. Using the modifications alluded to above, we may establish that $t_{((n,\alpha),(n,\alpha))}\ t_{((m,\beta),(m,\beta))}\subset t_{((n,\alpha),(n,\alpha))}$ if $n>m$ or $n=m$ and $\alpha\leqq\beta$ and $t_{((n,\alpha),(n,\alpha))}\ t_{((m,\beta),\ (m,\beta))}\subset t_{((m,\beta),(m,\beta))}$ if $m>n$ or $n=m$ and $\beta\leqq\alpha$ as in the proof of [4, lemma 9]. Let $g\in t_{((r,\beta),(r,\beta))}$ and $h\in t_{((r,\alpha),(r,\alpha))}$. Hence, $g\mathscr{L}f_{(r,\beta)}$ for some $f_{(r,\beta)}\in E_{(r,\beta)}$. Thus, $g\ e_{(r,\alpha)}\mathscr{L}f_{(r,\beta)}e_{(r,\alpha)}$. Since $f_{(r,\beta)}\ e_{(r,\alpha)}\in E_{(r,\alpha\wedge\beta)}$, $ge_{(r,\alpha)}\in D_{\alpha\wedge\beta}$. Hence, $ge_{(r,\alpha)}\in t_{((s,\beta\wedge\alpha),\ (r,\beta\wedge\alpha))}$ for some $s\in I^0$. Hence, $ge_{(r,\alpha)}=a^{-s}\ a^r\ z$ for some $z\in t_{((r,\alpha\wedge\beta),\ (r,\alpha\wedge\beta))}$ by lemma 5. Utilizing [3, lemma 1.5],

$$\begin{aligned}ga^{-(r+1)}&=g(e_{(r+1,\alpha_0)}\ a^{-(r+1)})\\ &=g\ e_{(r,\alpha_0)}\ e_{(r+1,\alpha_0)}\,a^{-(r+1)}\end{aligned}$$

$$=g\ e_{(r,\ \alpha_0)}\ a^{-(r+1)}$$
$$=a^{-s}\ a^{r}(z\ a^{-(r+1)})$$

Hence, by [3, lemma 1.5], lemma 3, and lemma 2, $ga^{-(r+1)}\ t_{((s+1,\alpha_0),(0,\alpha_0))}$. However, $ga^{-(r+1)}\in t_{((r+1,\alpha_0),(0,\alpha_0))}$ by lemma 3. Thus. $r=s$ and $ge_{(r,\alpha)}\in t_{((r,\beta\wedge\alpha),(r,\beta\wedge\alpha))}$. Hence, $gh=g(e_{(r,\alpha)}h)=(ge_{(r,\alpha)})h\in t_{(r,\beta\wedge\alpha),(r,\beta\wedge\alpha))}$ by the remark made at the beginning of this proof.

REMARK 2. By [3, lemma 1.12], each $t_{((k,\alpha),(k,\alpha))}$ is a right group and $t_{((k,\alpha),(k,\alpha))}\cong t_{((s,\alpha),(s,\alpha))}$ for all $s,k>0$. By lemma 2, $E_T=E_S$ where $T=\cup\{t_{((n,\alpha),(n,\alpha))}: n\in I^0 \text{and}\ \alpha\in Y\}$. For each $(r,s)\in I^0$, define $g\alpha_{(r,s)}=a^{-r}a^{s}ga^{-s}a^{r}$ for $g\in T$.

LEMMA 7. (cf. [4, lemma 10]) *$\alpha_{(r,s)}$ is an endomorphism of T and*

1) $t_{((k,\alpha),(k,\alpha))}\alpha_{(r,s)}\subset t_{((r,\alpha_0),(r,\alpha_0))}$ *if* $s>k$

2) $t_{((k,\alpha),(k,\alpha))}\alpha_{(r,s)}\subset t_{((k+r-s,\alpha),(k+r-s,\alpha))}$ *if* $k\geqq s$. *$\alpha_{(s,s)}$ is an inner right translation of T determined by an idempotent of* $t_{((s,\alpha_0),(s,\alpha_0))}$.

PROOF. With the proof of [4, lemma 10] as a guide, lemma 3, [3, lemma 1.5], lemma 2, lemma 1, remark 1, [3, lemma1. 13], lemma 4, and lemma 6 are employed.

LEMMA 8. $a^{-k}a^{p}a^{-r}a^{s}=a^{-(k+r-min(p,r))}a^{p+s-min(p,r)}$.

LEMMA 9. (cf. [4, lemma 12]). *If* $g_{p\alpha}\in t_{((p,\alpha),(p,\alpha))}$ *and* $h_{s\beta}\in t_{((s,\beta),(s,\beta)}(a^{-k}a^{p}g_{p\alpha})(a^{-r}a^{s}h_{s\beta})=a^{-(k+r-min(p,r))}a^{p+s-min(p,r)}g_{p\alpha}\alpha_{(s,r)}h_{s\beta}$, *where* $g_{p\alpha}\alpha_{(s,r)}h_{s\beta}\in t_{((s,\beta),(s,\beta))}$ *if* $r>p$; $t_{(p+s-r,\alpha),(p+s-r,\alpha))}$ *if* $p>r$; $t_{((s,\alpha\wedge\beta),(s,\alpha\wedge\beta))}$ *if* $p=r$.

PROOF. Employ lemmas 2, 6, 7, and 8.

If X is a semigroup, let $\varepsilon(X)$ denote the semigroup of endomorphisms of X (iteration).

LEMMA 10. (cf. [4, lemma 13]) *The mapping $(n,r)\rightarrow\alpha_{(n,r)}$ is an anti-homomorphism of C, the bicyclic semigroup into $\varepsilon(T)$. Lemma* 11 *(cf.* [4, *lemma* 14]). $S\cong\{((n,k),g_{k\alpha}): g_{k\alpha}\in t_{((k,\alpha),(k,\alpha))}, n, k\in I^0, \alpha\in Y\}$ *under the multiplication* $((n,k),g_{k\alpha})\ ((r,s),h_{s\beta})=((n,k)(r,s),\ (g_{k\alpha}\alpha_{(s,r)}h_{s\beta}))$ *where juxtaposition denotes multiplication in C and T.*

PROOF. By lemma 6 and lemma 9, $(a^{-n}a^{p}g_{p\alpha})\phi=((n,p),\ g_{p\alpha})$ is the required isomorphism.

Let T be a semigroup which is a union of a collection of pairwise disjoint groups

$\{T_{(k,\alpha)} : k \in I^0,\ \alpha \in Y$, a semilattice with greatest element $\alpha_0\}$ such that

$$T_{(k,\alpha)}\, T_{(r,\beta)} \subset \begin{cases} T_{(k,\alpha)} & \text{if } k > r \\ T_{(r,\beta)} & \text{if } r > k \\ T_{(r,\alpha\wedge\beta)} & \text{if } r = k \end{cases}$$

and such that E_T is a lexicographically ordered band. We call T an L-semilattice Y of right groups.

LEMMA 12. (cf. [4, lemma 15]). *Let* $T=\{T_{(s,\alpha)} : s \in I^0,\ \alpha \in Y$, *a semilattice with greatest element* $\alpha_0\}$ *be an L-semilattice Y of right groups and let C denote the bicyclic semigroup. Let* $(n,r)\to\alpha$ *denote an anti-homomorphism of C into* $\varepsilon(T)$ *such that for each* $s \in I^0$, $\alpha_{(s,s)}$ *is an inner right translation of T determined by an idempotent of* $T_{(s,\alpha_0)}$ *(i.e. for all* $g \in T$, $g\alpha_{(s,s)} = ge$ *for some* $e \in E_{T(s,\alpha_0)}$*) and for* k, $r, s \in I^0$

$$T_{(k,\alpha)}\alpha_{(r,s)} \subset \begin{cases} T_{(r,\alpha_0)} & \text{if } s > k \\ T_{(k+r-s,\alpha_0)} & \text{if } k \geqslant s. \end{cases}$$

Let $S=\{(n,k), g_{k\alpha}) : g_{k\alpha} \in T_{(k,\alpha)},\ k \in I^0,\ \alpha \in Y\}$ *under the multiplication* $((n,k), g_{k\alpha})\ ((r,s), h_{s\beta}) = ((n,k)(r,s), g_{k\alpha}\alpha_{(s,r)}h_{s\beta})$ *where juxtaposition denotes multiplication in C and T. Then, S is a simple L-regular semigroup.*

PROOF. Closure and associativity are immediate. The following facts are established as in the proof of [3, lemma 1.15] with the usual modifications: S is simple, $E_S=\{((k,k), g_{k\alpha}) : g_{k\alpha} \in E_{T_{(k,\alpha)}}, k \in I^0, \alpha \in Y\}$, $((n,k)g_{k\alpha})\mathscr{R}((r,s), h_{s\beta})$ iff $n=r$ and $\alpha=\beta$, and $((n,k), g_{k\alpha})\ \mathscr{D}((r,s), h_{s\beta})$ iff $\alpha=\beta$. Thus, we may write the $\mathscr{R}$-classes of E_S as $\{E_{(k,\alpha)} : k \in I^0, \alpha \in Y\}$ where $E_{(k,\alpha)} = \{((k,k), g_{k\alpha}) : g_{k\alpha} \in E_{T_{(k,\alpha)}}\}$. Thus, $E_{(k,\alpha)}$ and $E_{(r,\beta)}$ are contained in the same $\mathscr{D}$-class of S iff $\alpha=\beta$. By a straight forward calculation, E_S is a lexicographically ordered band. The $\mathscr{D}$-classes of S are $\{D\alpha : \alpha \in Y\}$ where $D_\alpha=\{((\mathrm{n},\mathrm{k}), \mathrm{g}_{k\alpha}) : \mathrm{g}_{k\alpha} \subset T_{(k,\alpha)}, n, k \in I^0\}$. Since each is a regular $\mathscr{D}$-class, S is a regular semigroup. Let $\beta_{(n,r)} = \alpha_{(r,n)}$.

THEOREM 1. *Let S be a simple L-regular semigroup. Then, there exists an L-semilattice Y of right groups* $T=\bigcup\{T_{(k,\alpha)} : k \in I^0, \alpha \in Y$, *a semilattice with greatest element* $\alpha_0\}$ *and a homomorphism* $(n,r)\to\beta_{(n,r)}$ *of C, the bicyclic semigroup into* $\varepsilon(T)$, *the semigroup of endomorphisms of T, such that*

(1) *for each* $k \in I^0$, *there exists* $e_{(k,\alpha_0)} \in E_{T(k,\alpha_0)}$ *such that* $g\beta_{(k,k)} = ge_{(k,\alpha_0)}$ *for all* $g \in T$;

(2) *For each* $k, r, s \in I^0$,

$$T_{(k,\alpha)}\ \beta_{(s,r)} \subset \begin{cases} T_{(r,\alpha_0)} & \text{if } s>k \\ T_{(k+r-s,\alpha)} & \text{if } k\geqslant s \end{cases}$$

Furthermore, $S \cong \{((n, k), g_{k\alpha}) : g_{k\alpha} \in T_{(k,\alpha)},\ n, k \in I^0,\ \alpha \in Y\}$ *under the multiplication*

(3) $((n,k), g_{k\alpha})((r,s), h_{s\beta}) = ((n,k)(r,s), g_{k\alpha}\ \beta_{(r,s)} h_{s\beta})$ *where juxtaposition denotes multiplication in C and T. Conversely, let T be an L-semilattice Y of right groups* $\{T_{(k,\alpha)} : k \in I^0, \alpha \in Y$, *a semilattice with greatest element* $\alpha_0\}$, *and let* $(n,r) \rightarrow \beta_{(n,r)}$ *be a homomorphism of C into* $\varepsilon(T)$ *such that* (1) *and* (2) *are valid. Then,* $((n,k), g_{k\alpha}) : g_{k\alpha} \in T_{(k,\alpha)}, n, k \in I^0,\ \alpha \in Y\}$ *under the multiplication* (3) *is a simple L-regular semigroup.*

PROOF. The theorem is valid by lemma 6, remark 2, lemma 7, lemma 10, lemma 11.

A semigroup T which is a union of a collection of pairwise disjoint right groups $\{T_k : k \in I^0\}$ such that $T_k T_r \subset T_{\max(k,r)}$ and E_T is a naturally ordered band is termed an E-chain of right groups. Let N denote the natural numbers.

COROLLARY 1. (Warne, [4]). *Let S be a simple E-regular semigroup. Then, there exists an E-chain T of right groups* $\{T_{kd+i} : k \in I^0, d \in N, i \in \{0, 1 \cdots. d-1\}$ *and a homomorphism* $(n,r) \rightarrow \beta_{(n,r)}$ *of C into* $\varepsilon(T)$, *such that*

(1) *for each* $k \in I^0$, *there exists* $e_{kd} \in E_{T_{kd}}$ *such that* $g\ \beta_{(k,k)} = g\ e_{kd}$ *for all* $g \in T$;

(2) *for each* $k, r, s \in I^0$,

$$T_{kd+i}\,\beta_{(s,r)} \subset \begin{cases} T_{rd} & \text{if } s>k \\ T_{(k+r-s)d+i} & \text{if } k\geqslant s \end{cases}$$

Furthermore, $S \cong \{((n,k), g_{ki}) : g_{ki} \in T_{kd+i}, n, k \in I^0,\ 0 \leqslant i < d\}$ *under the multiplication*

(3) $((n,k),\ g_{ki})\ ((r,s),\ h_{sj}) = ((n,k)\,(r,s),\ g_{ki}\ \beta_{(r,s)}\ h_{sj})$ *where juxtaposition denotes multiplication in C and T.*

Conversely, let T be an E-chain of right groups $\{T_{kd+i} : k \in I^0, d \in N, i \in \{0, 1, \cdots, d-1\}\}$ *and let* $(n,r) \rightarrow \beta_{(n,r)}$ *be a homomorphism of C into* $\varepsilon(T)$ *such that* (1) *and* (2) *are valid. Then,* $\{((n,k), g_{ki}) : g_{ki} \in T_{kd+i}, n, k \in I^0, d \in N, i \in \{0, 1, \cdots, d-1\}\}$ *under the multiplication*

(3) *is a simple E-regular semigroup.*

PROOF. Let S be a simple E-regular semigroup. Then, S has d $\mathscr{D}$-classes, $D_0\ D_1$

$\cdots, D_{d-1}$, say [4, lemma 1]. We may write $E(R)=\{E_{nd+i}=E_{(n,i)}: n\in I^0, i\in\{0,1, d-1\}\}$ with $E_{(n,i)}<E_{(m,j)}$ iff $n>m$ or $n=m$ and $i>j$. By [4, remark 1], $E_{(n,i)}$ and $E_{(n,j)}$ are contained in the same $\mathscr{D}$-class of S iff $i=j$. Thus, S is a simple L-regular semigroup with $Y=\{0>1>\cdots>d-1\}$. To obtain the corollary let $T_{(k,i)}=T_{kd+i}$ and $\alpha_0=0$ in theorem 1.

A semilattice E is termed lexicographically ordered if $E=\{e_{(n,\alpha)}: n\in I^0, \alpha\in Y$, a semilattice with greatest element$\}$ and $e_{(n,\alpha)}<e_{(m,\beta)}$ iff $n>m$ or $n=m$ and $\alpha<\beta$. A regular semigroup S is termed an L-inverse semigroup if E_S is a lexicographically ordered semilattice and $e_{(n,\alpha)}$ and $e_{(m,\beta)}$ are contained in the same $\mathscr{D}$-class of S iff $\alpha=\beta$. A semigroup T is termed an L-semilattice Y of groups if T is a union of a collection of pairwise disjoint groups $\{T_{(k,\alpha)}: k\in I^0, \alpha\in Y$, a semilattice with greatest element$\}$ and E_T is a lexicographically ordered semilattice.

COROLLARY 2. *Let S be a simple L-inverse semigroup. Then, there exists an L-semilattice Y of groups $T=\bigcup\{G_{(n,\alpha)}: n\in I^0, \alpha\in Y$, a semilattice with greatest element $\alpha_0\}$ and a homomorphism $(n,r)\to\beta_{(n,r)}$ of C into $\varepsilon(T)$ such that*

(1) *for each $k\in I^0$, there exists $e^2_{(k,\alpha_0)}=e_{(k,\alpha_0)}$ such that $g\,\beta_{(k,k)}=g\,e_{(k,\alpha_0)}$ for all $g\in T$;*

(2) *for each $k,r,s\in I^0$,*

$$G_{(k,\alpha)}\beta_{(s,r)}\subset\begin{cases}G_{(r,\alpha_0)} & \text{if } s>k\\ G_{(k+r-s,\alpha)} & \text{if } k\geqslant s\end{cases}$$

Furthermore, $S\cong\{((n,k), g_{k\alpha}): g_{k\alpha}\in G_{(k,\alpha)}, n,k\in I^0, \alpha\in Y\}$ under the multiplication

(3) $((n,k), g_{k\alpha})((r,s), h_{s\beta})=((n,k)(r,s), g_{k\alpha}\beta_{(r,s)}h_{s\beta})$ *where juxtaposition denotes multiplication in C and T.*

Conversely, let T be an L-semilattice Y of groups $\{G_{(k,\alpha)}: k\in I^0, \alpha\in Y$, a semilattice with greatest element $\alpha_0\}$ and let $(n,r)\to\beta_{(n,r)}$ be a homomorphism of C into $\varepsilon(T)$ such that (1) and (2) are valid. Then, $\{((n.k), g_{k\alpha}): g_{k\alpha}\in G_{(k,\alpha)}, n,k\in I^0, \alpha\in Y\}$ under the multiplication (3) is a simple L-inverse semigroup.

PROOF. Noting that each $t_{((n,\alpha),(n,\alpha))}$ is a group if S is an inverse semigroup, and utilizing [1, lemma 4.8] corollary 2 is an immediate consequence of theorem 1.

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