A NOTE ON HYPERSURFACES OF ALMOST CONTACT MANIFOLDS

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1. Introduction.

In a recent paper [1], the authors consider a $2n$-dimensional manifold $M$ imbedded in almost contact manifold $M^{2n+1}$ with fundamental affine collineation $\phi$, fundamental vector field $\xi$ and contact form $\eta$, and assume that for each $p \in M$ the vector field $\xi$ does not belong to the tangent hyperplane of the hypersurface. This means that the vector field $\xi$ can be taken as the "affine normal" to the hypersurface.

More recently [2], in the case which $\xi$ is always tangent to $M$, it is known that there exists a vector field $N$ playing the role of "affine normal" along the hypersurface.

In this paper, we consider the case where $\xi$ is always tangent to $M$.

2. Hypersurfaces of almost contact manifolds.

Let $\tilde{M}=\tilde{M}^{2n+1}(\phi, \xi, \eta)$ be an almost contact manifold, and let $M=M^{2n}$ be a hypersurface imbedded in $\tilde{M}$. Throughout this paper, we assume that the vector field $\xi$ is always tangent to $M$. Then it is known that a vector field $N$ exists along the hypersurface $M$ such that

\begin{align}
\phi N &= -A, \quad \eta(N) = 0 \\
\phi X &= fX + \alpha(X) \cdot N
\end{align}

for some vector field $A$ on $M$, $(1,1)$-type tensor field $f$ and 1-form $\alpha$.

Applying $\phi$ to the relation (2.1), we get

$$-X + \eta(X) \xi = f^2 X + \alpha(fX)N - \alpha(X)A,$$

which shows that

\begin{align}
f^2 &= -I + \eta \otimes \xi + \alpha \otimes A, \\
\alpha(fX) &= 0, \quad \eta(A) = 0, \quad \alpha(\xi) = 0, \quad \alpha(A) = 1, \\
f(\xi) &= 0, \quad f(A) = 0, \quad \eta(fX) = 0
\end{align}

for any $X \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the set of all vector fields on $M$. Thus we...
have that, in an almost contact manifold $\tilde{M}$, a hypersurface $M$ for the vector field $\xi$ to be tangent to $M$ admits $(f, \xi, A, \eta, \alpha, \lambda)$—structure.

Moreover, if we define a tensor field $\tilde{J}$ as
\[ \tilde{J} = f + \eta \otimes A, \]
then we obtain
\[ \tilde{J}^2(X) = f^2(X) + \eta(fX)A + \eta(X)f(A) + \eta(X)\eta(A)A = f^2(X), \]
that is, $\tilde{J}^2 = f^2$ on $M$. From which we have that
\[ \tilde{J}^4(X) = -\tilde{J}^2(X) \]
on $M$, by virtue of (2.2). Since $\tilde{J}$ has the same rank at each point of $M$, we find that the tensor field $\tilde{J}$ defined as (2.3) is a quartic structure in $M$.

On the other hand, for the same $\tilde{J}$ we get
\[ \tilde{J}^2 = -I + \eta \otimes \xi + \alpha \otimes A, \]
and $\eta(A) = 0$, $\alpha(\xi) = 0$.

Hence we can see that the hypersurface $M$ is to be globally framed.

Combining the above results.

**Theorem 1.** The hypersurface $M$ imbedded in almost contact manifold $\tilde{M}$ in such a way that the vector field $\xi$ is always tangent to $M$ is a globally framed quartic manifold.

3. Hypersurfaces of Sasakian manifolds.

Let $\tilde{M} = \mathbb{R}^{2n+1}(\phi, \xi, \eta, \mathbb{R})$ be an almost contact manifold, and let $\nabla$ be the Riemannian connection of $\tilde{g}$. For $X, X \in \mathcal{X}(M)$, we get
\[ \nabla_X Y = \nabla_X Y + h(X, Y)N, \]
\[ \nabla_X N = -HX + \omega(X)N, \]
where $\nabla_X Y$ and $-HX$ are the tangential parts (with respect to $N$) of $\nabla_X Y$ and $\nabla_X N$, respectively, to $M$. We can see that $\nabla : (X, Y) \rightarrow \nabla_X Y$ is a symmetric connection on $M$, $h$ is symmetric, and is called the second fundamental form of $M$ (with respect to $N$).

If $h = 0$ on $M$, then $M$ is called to be totally geodesic. Let $g$ be the induced metric: $g = \tilde{g}/M$. In general, the connection $\nabla$ is not the Levi-Civita connection of $g$. Using (3.1) and (3.2), we obtain
\[ (\nabla_X g)(Y, Z) = h(X, Y)g(N, Z) + h(X, Z)g(Y, N) \]

Suppose that $\nabla$ is the Levi-Civita connection of the induced metric $g$, then we find
\[ 2g(\nabla_X Y, Z) = X \cdot g(Y, Z) + Y \cdot g(X, Z) - Z \cdot g(X, Y) + g([X, Y], Z) \]
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From which we have

\[
(\mathcal{X}) = \mathcal{X}(M).
\]

Since \( N \) is an affine normal, we find from (3.3) and (3.4) the following:

**THEOREM 2.** In order that the connection \( \nabla \) of the hypersurface \( M \) imbedded in almost contact manifold \( \tilde{M} \) in such a way that the vector field \( \xi \) is always tangent to \( M \) is a Riemannian connection of \( g = \bar{g}/M \), it is necessary and sufficient that \( M \) is totally geodesic.

Now, we assume that \( \tilde{M} = \tilde{M}^{2n+1}(\phi, \xi, \eta, \bar{g}) \) is a Sasakian manifold: that is, the following holds good:

\[
(U_{\phi}) = \mathcal{X}(\tilde{M}),
\]

where \( \mathcal{X}(\tilde{M}) \) is the set of all vector fields on \( \tilde{M} \). It is well known that (3.5) implies

\[
(U_{\xi}) = \mathcal{X}(\tilde{M}),
\]

Summing up theorem 2 and 3, we obtain

**THEOREM 4.** The induced connection \( \nabla \) of the hypersurface \( M \) imbedded in a Sasakian manifold \( \tilde{M} \) in such a way that the vector field \( \xi \) is always tangent to \( M \) cannot be a Riemannian connection of \( g = \bar{g}/M \).

4. Hypersurfaces of affinely cosymplectic manifold.

We assume that \( \tilde{M} = \tilde{M}^{2n+1}(\phi, \xi, \eta) \) is affinely cosymplectic; an almost contact manifold \( \tilde{M}^{2n+1}(\phi, \xi, \eta) \) with a symmetric affine connection \( \tilde{\nabla} \) satisfies

\[\tilde{\nabla}\phi = 0, \tilde{\nabla}\eta = 0.\]

Then we have

\[
(\mathcal{X}_X) = \mathcal{X}(M),
\]

On the other hand
\[ v_X Y = (V_X f) Y + f(V_X Y) + h(X, fY) N - \alpha(Y) H X + \alpha(Y) \omega(X) N + (V_X \alpha Y) N. \]

Comparing (4.2) and (4.3), we get

\[ (V_X f) Y = \alpha(Y) H X - h(X, Y) A, \]
\[ (V_X \alpha) Y = -h(X, fY) - \alpha(Y) \omega(X). \]

Moreover, we find from (4.1), (4.4) and (4.5)

\[ [f, f] (X, Y) + d \eta(X, Y) \xi + d\alpha(X, Y) A = \alpha(Y) H f X - \alpha(X) H f Y + \alpha(X) f H Y - \alpha(Y) f H X + (\alpha \wedge \omega)(X, Y) A \]

Now we assume that \( M \) is to be totally flat, then \( H X = 0 \).

Thus we find from (4.6) the following:

**THEOREM 5.** Suppose that the hypersurface \( M \) imbedded in an affinely cosymplectic manifold \( \tilde{M} \) in such a way that the vector field \( \xi \) is always tangent to \( M \) is to be totally flat. Then the necessary and sufficient condition in order that \( (f, \xi, A, \eta, \alpha, \lambda) \)-structure is normal is \( \alpha \wedge \omega = 0 \) on \( M \).

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**REFERENCE**
