SOME RELATIONS BETWEEN MEIJER-LAPLACE AND HANKEL TRANSFORMS OF TWO VARIABLES

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1. Introduction.

The author has introduced the Meijer-Laplace transform of two variables [3, 4] in the form

$$(1.1)\quad F(p,\ q)=pq\int_0^\infty\int_0^\infty G^{m+1,\,0}_{m,\,m+1}\left(px\,\middle|\,\begin{matrix}a_1+b_1,\ \cdots,\ a_m+b_m\\ a_1,\ \cdots\cdots\cdots,\ a_{m+1}\end{matrix}\right)G^{n+1,\,0}_{n,\,n+1}\left(qx\,\middle|\,\begin{matrix}c_1+d_1,\cdots,c_n+d_n\\ c_1,\ \cdots\cdots\cdots,\ c_{n+1}\end{matrix}\right)$$
$$\times f(x,\ y)dx\ dy,\ R(p,\ q)>0$$

which is generalisation of the well known Laplace transform of two variables [2, p. 657]

$$(1.2)\quad F(p,\ q)=pq\int_0^\infty\int_0^\infty e^{-px-qy}f(x,\ y)dxdy,\ R(p,\ q)>0.$$

Setting $b_i=0,\ i=1,\ 2,\cdots\cdots,\ m-1:d_j=0,\ j=1,\cdots\cdots,\ n-1:$

(A) and $b_m=0,\ a_{m+1}=0,\ d_n=0,\ c_{n+1}=0$, (1.1) reduces to (1.2);

(B) and $a_{m+1}=b_m=-m_1-k_1,\quad a_m=m_1-k_1,\quad c_{n+1}=d_n=-m_2-k_2,\quad c_n=m_2=k_2,$ (1.1) reduces to Meijer transform of two variables [6, p. 83]

$$(1.3)\quad F(p,\ q)=pq\int_0^\infty\int_0^\infty(px)^{-k_1-\frac{1}{2}}(qy)^{-k_2-\frac{1}{2}}e^{-\frac{1}{2}px-\frac{1}{2}qy}W_{k_1+\frac{1}{2},\ m_1}(px)W_{k_2+\frac{1}{2},\,m_2}$$
$$\times(qy)f(x,\ y)dxdy,\ R(p,\ q)>0;$$

(C) and $a_m=2m_1,\ a_{m+1}=0,\ b_m=\frac{1}{2}-m_1-k_1;\ c_n=2m_2,\ c_{n+1}=0,\ d_n=\frac{1}{2}-m_2-k_2,$ (1.1) reduces to [7, p. 49]

$$(1.4)\quad F(p,\ q)=pq\int_0^\infty\int_0^\infty(px)^{m_1-\frac{1}{2}}(qy)^{m_2-\frac{1}{2}}e^{-\frac{1}{2}px-\frac{1}{2}qy}W_{k_1,\,m_1}(px)W_{k_2,\ m_2}(qy)$$
$$\times f(x,\ y)dxdy,\ R(p,\ q)>0,$$

and shall call it Varma transform of two variables.

We shall denote (1.1), (1.2), (1.3) and (1.4) symbolically

$$F(p,\ q)=G[f(x,\ y)],\ F(p,\ q)\doteqdot f(x,\ y),\ F(p,\ q)\xleftarrow[m_1,\ m_2]{k_1+\frac{1}{2},\ k_2+\frac{1}{2}}f(x,\ y)$$

and $\qquad F(p,\ q) \xleftarrow[m_1,\, m_2]{k_1,\, k_2} f(x,\ y)$ respectively.

The Hankel transform of order n_1 and n_2 of a function $f(x,\ y)$ is given by [5, p. 47]

$$(1.5)\qquad F(p,\ q)=\int_0^\infty\int_0^\infty \sqrt{pqxy}\, J_{n_1}(px)J_{n_2}(qy)f(x,\ y)dxdy,\ R(n_1,\ n_2)>-1,$$

and denote it symbolically as

$$F(p,\ q)=H_{n_1, n_2}[f(x,y) : p,\ q].$$

In what follows the symbols (a_r), $\Delta(n,\ a)$, $\Delta(n,\ \pm a)$ and $\Delta((n,\ a_r))$ have been employed to denote the set of parameters $a_1,\ a_2,\ \cdots\cdots,\ a_r$: $\frac{a}{n},\ \frac{a+1}{n},\ \cdots\cdots,\ \frac{a+n-1}{n}$; $\Delta(n,\ a)$, $\Delta(n,\ -a)$ and $\Delta(n,\ a_1)$, $\Delta(n,\ a_2), \cdots\cdots, \Delta(n,\ a_r)$ respectively, throughout this paper.

2. The following formula [8, p. 401] will be required in the sequel.

$$(2.1)\qquad \int_0^\infty t^{\sigma-1} G^{h,\ l}_{q,\ r}\left(pt\,\Big|\,\begin{matrix}(\alpha_q)\\(\beta_r)\end{matrix}\right) G^{\alpha,\ \beta}_{\gamma,\ \delta}\left(zt^{n/s}\,\Big|\,\begin{matrix}(a_\nu)\\(b_\delta)\end{matrix}\right)dt$$

$$=(2\pi)^{(1-s)\left(\alpha+\beta-\frac{1}{2}\gamma-\frac{1}{2}\delta\right)+(1-n)\left(h+l-\frac{1}{2}q-\frac{1}{2}r\right)}$$

$$\times S^{\sum_{i=1}^{\delta} b_i-\sum_{j=1}^{r} a_j+\frac{1}{2}\gamma-\frac{1}{2}\delta+1}\, n^{\sum_{i=1}^{r}\beta_i-\sum_{j=1}^{q}\alpha_j+\left(\sigma-\frac{1}{2}\right)(r-q)}$$

$$\times p^{-\sigma} G^{\alpha s+nl,\ s\beta+nh}_{s\gamma+nr,\ s\delta+nq}\left(\frac{z^s s^{s(\gamma-\delta)}}{p^n n^{n(q-r)}}\,\Bigg|\,\begin{matrix}\Delta((s,a_\beta)),\ \Delta((n,-\beta_r-\sigma+1)),\ \Delta(s,a_{\beta+1}),\ \cdots\cdots,\ \Delta(s,a_r)\\ \Delta((s,b_\alpha)),\ \Delta((n,-\alpha_q-\sigma+1)),\ \Delta(s,b_{\alpha+1}),\ \cdots\cdots,\ \Delta(s,b_\delta)\end{matrix}\right),$$

where

$0\leqslant h\leqslant r,\ \ 0\leqslant l\leqslant q,\ \ 0\leqslant\alpha\leqslant\delta,\ \ 0\leqslant\beta\leqslant\gamma,\ \ 2(\alpha+\beta)>\gamma+\delta,\ \ 2(h+l)>q+\gamma,\ \ |\arg z|<\left(\alpha+\beta-\frac{1}{2}\gamma-\frac{1}{2}\delta\right)\pi,\ \ R\left(\min\beta_i+\frac{n}{s}.\ \ \min b_j\right)>R(-\sigma)>R\left(\frac{n}{s}a_f+\alpha_g-\frac{n}{s}-1\right)$

$$i=1, 2, \cdots\cdots, h\ ;\ j=1, 2, \cdots\cdots, \alpha\ ;\ g=1, 2, \cdots\cdots, l \text{ and } f=1, 2, \cdots\cdots,\ \beta.$$

3. THEOREM 1.

If

$$(3.1)\qquad h(p,\ q)=G[f(x,\ y)],$$

and

$$(3.2)\qquad F(p,\ q)=H_{n_1, n_2}\left[(xy)^u h(x^{\frac{r_1}{s_1}},\ y^{\frac{r_2}{s_2}}) : p,\ q\right],$$

then

$$(3.3)\quad F(p,\ q)=\frac{2^{2u+\frac{r_1}{s_1}+\frac{r_2}{s_2}+1}}{b^{u+\frac{r_1}{s_1}+1}\,q^{u+\frac{r_2}{s_2}+1}}\int_0^\infty\int_0^\infty \phi(x,\ y,\ p,\ q)f(x,\ y)dxdy$$

where

$$\phi(x,y,p,q)=(2\pi)^{1-s_1-s_2}r_1^{u+\frac{r_1}{s_1}+\frac{1}{2}}r_2^{u+\frac{r_2}{s_2}+\frac{1}{2}}(2s_1)^{a_{m+1}-\sum_{i=1}^{m}b_i+\frac{1}{2}}(2s_2)^{c_{n+1}-\sum_{j=1}^{n}d_j+\frac{1}{2}}$$

$$\times G^{2s_1(m+1),\ r_1}_{2s_1m+2r_1,\ 2s_1m+2s_1}\left(\frac{x^{2s_1}(2r_1)^{2r_1}}{p^{2r_1}(2s_1)^{2s_1}}\,\middle|\,\begin{matrix}\Delta\left(r_1,\ \mp\frac{1}{2}n_1-\frac{1}{2}\ \frac{r_1}{s_1}-\frac{u}{2}+\frac{1}{4}\right),\Delta((2s_1,\ a_m+b_m))\\ \Delta((2s_1,\ a_{m+1}))\end{matrix}\right)$$

$$\times G^{2s_2(n+1),\ r_2}_{2s_2n+2r_2,\ 2s_2(n+1)}\left(\frac{y^{2s_1}(2r_1)^{2r_2}}{q^{2r_2}(2s_2)^{2s_2}}\,\middle|\,\begin{matrix}\Delta\left(r_2,\ \mp\frac{1}{2}n_2-\frac{1}{2}\frac{r_2}{s_2}-\frac{u}{2}+\frac{1}{4}\right),\Delta((2s_2,\ c_n+d_n))\\ \Delta((2s_2,\ c_{n+1}))\end{matrix}\right),$$

the Meijer Laplace transform of $|f(x,\ y)|$ *and Hankel transform of* $|(xy)^u,\ h(x^{\frac{r_1}{s_1}},\ y^{\frac{r_2}{s_2}})|$ *exist,* (3.3) *is absolutely convergent;* $r_1,\ r_2,\ s_1,\ s_2$ *are all positive integers and* $R\left(u+n_1+\frac{r_1}{s_1}+\frac{r_1}{s_1}a_j\right)>-\frac{3}{2},\ j=1,\cdots\cdots,\ m+1;\ p>0,\ q>0,\ R(n_1,\ n_2)>-1,$ $R\left(u+n_2+\frac{r_2}{s_2}+\frac{r_2}{s_2}c_i\right)>-\frac{3}{2},\quad i=1,\cdots\cdots,\ n+1.$

PROOF. Substituting the value of $h(s^{\frac{r_1}{s_1}},\ t^{\frac{r_2}{s_2}})$ from (1.1) in

$$F(p,\ q)=\int_0^\infty\int_0^\infty\sqrt{pqst}\,J_{n_1}(ps)J_{n_2}(qt)(st)^u h(s^{\frac{r_1}{s_1}},\ t^{\frac{r_2}{s_2}})dsdt,$$

we have

$$(3.4)\quad F(p,\ q)=\int_0^\infty\int_0^\infty\sqrt{pqst}\,J_{n_1}(ps)J_{n_2}(qt)s^{u+\frac{r_1}{s_1}}t^{u+\frac{r_2}{s_2}}$$

$$\times\left[\int_0^\infty\int_0^\infty G^{m+1,\ 0}_{m,\ m+1}\left(s^{\frac{r_1}{s_1}}x\,\middle|\,\begin{matrix}(a_m+b_m)\\(a_{m+1})\end{matrix}\right)G^{n+1,\ 0}_{n,\ n+1}\left(t^{\frac{r_2}{s_2}}y\,\middle|\,\begin{matrix}(c_n+d_n)\\(c_{n+1})\end{matrix}\right)f(x,y)dxdy\right]dsdt.$$

Changing the order of integration, using the identity [1, p. 219]

$$(3.5)\qquad J_n(x)=G^{1,\ 0}_{0,\ 2}\left(\frac{x^2}{4}\,\middle|\,\begin{matrix}-\\ \frac{n}{2},\ -\frac{n}{2}\end{matrix}\right),$$

substituting $s^2=S,\ t^2=T$ and evaluating the inner double integral by (2.1), the result (3.3) follows.

The change in the order of integration is justified as we observe that

(i) s, t-integral is absolutely convergent, if

$R\left(u+\frac{r_1}{s_1}+n_1+\frac{r_1}{s_1}a_i\right)>-\frac{3}{2}$, $i=1, 2, \cdots\cdots, m+1$, $p>0$, $q>0$, $R(n_1, n_2)>-1$,

$R\left(u+\frac{r_2}{s_2}+n_2+\frac{r_2}{s_2}c_j\right)>-\frac{3}{2}$, $j=1, 2, \cdots\cdots, n+1$.

(ii) x, y-integral is so, provided

$R(a_i+k_1)>-1$, $i=1, \cdots\cdots, m+1$: $R(c_j+k_2)>-1$, $j=1, 2, \cdots\cdots, n+1$, where $f(x, y)=0(x^{k_1}, y^{k_2})$ for small x and y.

For large values of x and y the G-functions involved vanish exponentially [1, p. 212, (11)]

(iii) and one of the repeated double integral is absolutely convergent as the integral in (3.3) exists.

Particular case:

(a) Taking $b_i=0$, $i=1, \cdots\cdots, m$, $a_{m+1}=0$; $d_j=0$, $j=1, \cdots\cdots, n$, $c_{n+1}=0$ in the theorem, we get:

If

$$h(p, q) \doteqdot f(x, y)$$

and

$$F(p, q)=H_{n_1, n_2}\left[(xy)^u h(x^{\frac{r_1}{s_1}}, y^{\frac{r_2}{s_2}}) : p, q\right],$$

then

$$(3.6) \quad F(p, q)=\frac{2^{2u+\frac{r_1}{s_1}+\frac{r_2}{s_2}+2}}{p^{u+\frac{r_1}{s_1}+1} q^{u+\frac{r_2}{s_2}+1}} \int_0^\infty\int_0^\infty \phi(x, y, p, q) f(x, y)\, dx dy$$

where

$$\phi(x, y, p, q)=(2\pi)^{1-s_1-s_2} r_1^{u+\frac{r_1}{s_1}+\frac{1}{2}} r_2^{u+\frac{r_2}{s_2}+\frac{1}{2}} (s_1 s_2)^{\frac{1}{2}}$$
$$\times G^{2s_1, r_1}_{2r_1, 2s_1}\left(\left(\frac{x}{2s_1}\right)^{2s_1}\left(\frac{2r_1}{p}\right)^{2r_1}\middle|\begin{matrix}\Delta\left(r_1, \mp\frac{1}{2}n_1-\frac{r_1}{2s_1}-\frac{u}{2}+\frac{1}{u}\right)\\ \Delta(2s_1, 0)\end{matrix}\right)$$
$$\times G^{2s_2, r_2}_{2r_2, 2s_2}\left(\left(\frac{y}{2s_2}\right)^{2s_2}\left(\frac{2r_2}{q}\right)^{2r_2}\middle|\begin{matrix}\Delta\left(r_2, \mp\frac{1}{2}n_2-\frac{r_1}{2s_1}-\frac{u}{2}+\frac{1}{u}\right)\\ \Delta(2s_2, 0)\end{matrix}\right)$$

the Laplace transform of $|f(x, y)|$ and Hankel transform of $|(xy)^u h(x^{\frac{r_1}{s_1}}, y^{\frac{r_2}{s_2}})|$

exist, the integral in (3.6) is absolutely convergent; r_1, r_2, s_1, s_2 are positive integers and $R\left(u+n_1+\frac{r_1}{s_1}\right)>-\frac{3}{2}$, $R\left(u+n_2+\frac{r_2}{s_2}\right)>-\frac{3}{2}$, $R(n_1, n_2)>-1$, $p>0$, $q>0$.

4. THEOREM 2.

If

$$R(p, q)=G\, f(x, y),$$

and

$$F(p, q)=H_{n_1, n_2}\left[(xy)^N h\left(x^{-\frac{r_1}{s_1}}, y^{-\frac{r_2}{s_2}}\right); p, q\right]$$

then

$$\text{(4.1)}\quad F(p, q)=\frac{2^{2N-\frac{r_1}{s_1}-\frac{r_2}{s_1}+1}}{p^{N-\frac{r_1}{s_1}+1}\, q^{N-\frac{r_2}{s_2}+1}}\int_0^\infty\int_0^\infty \phi(p, q, x, y)f(x, y)\,dx\,dy,$$

where

$$\phi(p,q,x,y)=(2\pi)^{1-r_1-r_2}s_1^{N-\frac{r_1}{s_1}+\frac{1}{2}}s_2^{N-\frac{r_2}{s_2}+\frac{1}{2}}(2r_1)^{a_{m+1}-\sum_{i=1}^{m}b+\frac{1}{2}}(2r_2)^{c_{n+1}-\sum_{i=1}^{n}d_i+\frac{1}{2}}$$

$$\times G^{2r_1m+2r\ +s_1,\,0}_{2r_1m,\,2r_1m+2r_1+2s_1}\left(\left(\frac{x}{2r_1}\right)^{2r_1}\left(\frac{p}{2s_1}\right)^{2s_1}\middle|\begin{matrix}\Delta((2r_1, a_m+b_m))\\ \Delta((2r_1, a_{m+1})), \Delta\left(s_1, \pm\frac{1}{2}n_1+\frac{1}{2}N-\frac{1}{2}\ \frac{r_1}{s_1}+\frac{3}{u}\right)\end{matrix}\right)$$

$$\times G^{2r_2n+2r_2+s_2,\,0}_{2r_2n,\,2r_2n+2r_2+2s_2}\left(\left(\frac{y}{2r_2}\right)^{2r_2}\left(\frac{q}{2s_2}\right)^{2s_2}\middle|\begin{matrix}\Delta((2r_2, c_n+d_n))\\ \Delta((2r_2, c_n+1)), \Delta\left(s_2, \pm\frac{1}{2}n_2+\frac{1}{2}N-\frac{1}{2}\ \frac{r_2}{s_2}+\frac{3}{u}\right)\end{matrix}\right),$$

the Meijer Laplace transform of $|f(x,y)|$ *and Hankel transform of* $|(xy)^N h(x^{-\frac{r_1}{s_1}}, y^{-\frac{r_2}{s_2}})|$ *exist, the integral in* (4.1) *is absolutely convergent,* r_1, r_2, s_1 *and* s_2 *are positive integers and*

$$R(n_1, n_2)>-1,\ R\left(N-\frac{r_1}{s_1}-\frac{r_1}{s_1}a_j+1\right)<0,\ j=1, \cdots\cdots, m+1.$$

$$R\left(N-\frac{r_2}{s_2}-\frac{r_2}{s_2}c_i+1\right)<0,\ i=1, 2, \cdots\cdots, n+1,\ p>0,\ q>0.$$

PROOF. Substituting the value of $h(s^{-\frac{r_1}{s_1}}, t^{-\frac{r_2}{s_2}})$ from (1.1) in

$$\text{(4.2)}\quad F(p, q)=\int_0^\infty\int_0^\infty \sqrt{pqst}\, J_{n_1}(ps)J_{n_2}(qt)(st)^N h(s^{-\frac{r_1}{s_1}}, t^{-\frac{r_2}{s_2}})\,ds\,dt,$$

we have

$$(4.3)\quad F(p,\ q)=\sqrt{pq}\int_0^\infty\int_0^\infty s^{N-\frac{r_1}{s_1}+\frac{1}{2}}t^{N-\frac{r_2}{s_2}+\frac{1}{2}}J_{n_1}(ps)J_{n_2}(qt)$$

$$\times\left[\iint G_{m,\ m+1}^{m+1,\ 0}\left(s^{-\frac{r_1}{s_1}}x\ \middle|\ \begin{matrix}(a_m+b_m)\\(a_{m+1})\end{matrix}\right)G_{n,n+1}^{n+1,0}\left(t^{-\frac{r_2}{s_2}}y\ \middle|\ \begin{matrix}(c_n+d_n)\\(c_{n+1})\end{matrix}\right)f(x,\ y)dxdy\right]dsdt.$$

Interchanging the order of integration and using the identity (3.5), we have

$$(4.4)\quad F(p,\ q)=\sqrt{pq}\int_0^\infty\int_0^\infty f(x,\ y)dxdy\left[\int_0^\infty\int_0^\infty G_{0,2}^{1,0}\left(\frac{p^2s^2}{4}\ \middle|\ \begin{matrix}-\\\frac{n_1}{2},\ -\frac{n_1}{2}\end{matrix}\right)G_{0,2}^{1,0}\left(\frac{q^2t^2}{4}\ \middle|\ \begin{matrix}-\\\frac{n_2}{2},\ -\frac{n_2}{2}\end{matrix}\right)\right.$$

$$\left.\times s^{N-\frac{r_1}{s_1}+\frac{1}{2}}t^{N-\frac{r_2}{s_2}+\frac{1}{2}}G_{m,m+1}^{m+1,\ 0}\left(s^{-\frac{r_1}{s_1}}x\ \middle|\ \begin{matrix}(a_m+b_m)\\(a_{m+1})\end{matrix}\right)G_{n,n+1}^{n+1,0}\left(t^{-\frac{r_2}{s_2}}y\ \middle|\ \begin{matrix}(c_n+d_n)\\(c_{n+1})\end{matrix}\right)dsdt\right].$$

Subituting $s^2=S$, $t^2=T$, using the identity [1, p. 209, (9)] and evaluating the inner double integral with the help of (2.1), the result follows.

The change in the order of integration is justified as we observe that:

(i) the s, t-integral is absolutely convergent, if

$$R\left(N-\frac{r_1}{s_1}-\frac{r_1}{s_1}a_j+1\right)<0,\ j=1,\ 2,\ \cdots\cdots,\ m+1,\ R(n_1,\ n_2)>-1,$$

$$R\left(N-\frac{r_2}{s_2}-\frac{r_2}{s_2}c_i+1\right)<0,\ i=1,\ 2,\ \cdots\cdots,\ n+1,\ p>0,\ q>0.$$

For small values of s and t, $G_{m,m+1}^{m+1,\ 0}\left(\frac{1}{s}\ \middle|\ \begin{matrix}(a_m)\\(b_{m+1})\end{matrix}\right)$, $G_{n,n+1}^{n+1,0}\left(\frac{1}{t}\ \middle|\ \begin{matrix}(a_n)\\(b_{n+1})\end{matrix}\right)$ vanishes exponentially [1, p. 212, (11)].

(ii) the x,y-integral is absolutely convergent, provided

$$R(a_j+k_1+1)>0,\ j=1,\ \cdots\cdots,\ m+1\ ;\ R\ (c_i+k_2+1)>0,$$

where $f(x,\ y)=0(x^{k_1},\ y^{k_2})$ for small x and y.

For large values of x and y, the G-function involved vanishes exponentially [1, p. 212, (11)].

(iii) one of the repeated double integral is absolutely convergent as the integral in (4.1) exist.

(a) Particular Case:

Setting $b_i=0$, $i=1,\ 2,\cdots\cdots,\ m$, $a_{m+1}=0$; $d_j=0$, $j=1,\ \cdots\cdots,\ n$, $c_{n+1}=0$, in the theorem 2, we obtain:

If

$$h(p,\ q)\doteqdot f(x,\ y),$$

and

$$F(p,\ q)=H_{n_1, n_2}\left[(xy)^N h(x^{-\frac{r_1}{s_1}},\ y^{-\frac{r_2}{s_2}}) : p,\ q\right]$$

then

$$(4.5)\qquad F(p,\ q)=\frac{2^{2N-\frac{r_1}{s_1}-\frac{r_2}{s_2}}}{p^{N-\frac{r_1}{s_1}+1}\, q^{N-\frac{r_2}{s_2}+1}}\int_0^\infty\int_0^\infty \phi(p,\ q,\ x,\ y)f(x,\ y)dxdy,$$

where

$$\phi(p,\ q,\ x,\ y)=(2\pi)^{1-r_1-r_2} s_1^{N-\frac{r_1}{s_1}+\frac{1}{2}} s_2^{N-\frac{r_2}{s_2}+\frac{1}{2}} \sqrt{4r_1 r_2}$$

$$\times G^{2r_1+s_1, 0}_{0, 2r_1+2s_1}\left(\left(\frac{x}{2r_1}\right)^{2r_1}\left(\frac{p}{2s_1}\right)^{2s_1}\Bigg|\, {\overline{\quad}\atop \Delta(2r_1, 0),\ \Delta\left(s_1,\ \pm\frac{n_1}{2}+\frac{N}{2}-\frac{r_1}{2s_1}+\frac{3}{u}\right)}\right)$$

$$\times G^{2r_2+s_2, 0}_{0, 2r_2+2s_2}\left(\left(\frac{y}{2r_2}\right)^{2r_2}\left(\frac{q}{2s_2}\right)^{2s_2}\Bigg|\, {\overline{\quad}\atop \Delta(2r_2, 0),\ \Delta\left(s_2,\ \pm\frac{n_2}{2}+\frac{N}{2}-\frac{r_2}{2s_2}+\frac{3}{u}\right)}\right),$$

the Laplace transform of $|f(x,\ y)|$ and Hankel transform of $|(xy)^N h(x^{-\frac{r_1}{s_1}},\ y^{-\frac{r_2}{s_2}})|$ exist, the integral in (4.5) is absolutely convergent, r_1, r_2, s_1 and s_2 are positive integers and

$$R(n_1,\ n_2)>-1,\quad R\left(N-\frac{r_i}{s_i}+1\right)<0,\ i=1,\ 2,\ p>0,\ q>0.$$

5. THEOREM 3.

If

$$h(p,\ q)=H_{n_1,\ n_2}[f(x,\ y) : p,\ q],$$

and

$$F(p,\ q)=G\left[(xy)^N h(x^{\frac{r_1}{s_1}},\ y^{\frac{r_2}{s_2}})\right],$$

then

$$(5.1)\quad F(p,\ q)=\frac{(2\pi)^{1-r_1-r_2}}{p^{N+\frac{r_1}{2s_2}}\, q^{N+\frac{r_2}{2s_2}}}\int_0^\infty\int_0^\infty \sqrt{st}\,\phi(p,\ q,\ s,\ t)f(s,\ t)\ dsdt,$$

where $\phi(p,\ q,\ s,\ t)=(2r_1)^{a_{m+1}-\sum_{i=1}^{m}b_i+N+\frac{r_1}{2s_1}+\frac{1}{2}}(2r_2)^{c_{n+1}-\sum_{j=1}^{n}d_j+N+\frac{r_2}{2s_2}+\frac{1}{2}}$

$$\times G^{s_1,\,2r_1m+2r_1}_{2r_1m+2r_1,\ 2s_1+2r_1m}\left(\left(\frac{2r_1}{p}\right)^{2r_1}\left(\frac{s}{2s_1}\right)^{2s_1}\middle|\begin{matrix}\Delta\left(\left(2r_1,\ -a_{m+1}-N-\frac{r_1}{2s_1}\right)\right)\\ \Delta\left(s_1,\pm\frac{n_1}{2}\right),\Delta\left(\left(2r_1,-a_m-b_m-N-\frac{r_1}{2s_1}\right)\right)\end{matrix}\right)$$

$$\times G^{s_2,\,2r_2n+2r_2}_{2r_2n+2r_2,\ 2s_2+2r_2n}\left(\left(\frac{2r_2}{q}\right)^{2r_2}\left(\frac{t}{2s_2}\right)^{2s_2}\middle|\begin{matrix}\Delta\left(\left(2r_2,\ -c_{n+1}-N-\frac{r_2}{2s_2}\right)\right)\\ \Delta\left(s_2,\pm\frac{n_2}{2}\right),\ \Delta\left(\left(2r_2,-c_n-d_n-N-\frac{r_2}{2s_2}\right)\right)\end{matrix}\right),$$

the Hankel transform of $|f(x,\ y)|$ *and Meijer-Laplace transform of* $|(xy)^N h(x^{\frac{r_1}{s_1}},\ y^{\frac{r_2}{s_2}})|$ *exist, the integral in* (5.1) *is absolutely convergent, and*

$$R\left(N+\frac{r_1}{2s_1}+\frac{r_1n_1}{s_1}+a_i+1\right)>0,\ i=1,\ \cdots\cdots,\ m+1,\ R(p)>0,\ |\arg p|<\frac{\pi}{2},$$

$$R\left(N+\frac{r_2}{2s_2}+\frac{r_2n_2}{s_2}+c_j+1\right)>0,\ j=1,\ \cdots\cdots,\ n+1,\ R(q)>0,\ |\arg q|<\frac{\pi}{2}.$$

PROOF. It is given that

$$(5.2)\quad F(p,\ q)=pq\int_0^\infty\int_0^\infty G^{m+1,\,0}_{m,\,m+1}\left(px\middle|\begin{matrix}(a_m+b_m)\\(a_{m+1})\end{matrix}\right)G^{n+1,\,0}_{n,\,n+1}\left(qy\middle|\begin{matrix}(c_n+d_n)\\(c_{n+1})\end{matrix}\right)$$
$$\times(xy)^N h(x^{\frac{r_1}{s_1}},\ y^{\frac{r_2}{s_2}})dxdy.$$

Substituting the value of $h(x^{\frac{r_1}{s_1}},\ y^{\frac{r_2}{s_2}})$ from (1.5) in (5.2), changing the order of integration which is justified under the conditions stated in the theorem, using the identity (3.5) and then evaluating the inner double with the help of (2.1), the result (5.1) is obtained.

(a) Particular case:

Taking $b_i=0$, $i=1$, ……, m, $a_{m+1}=0$: $d_j=0$, $j=1$, ……, n, $c_{n+1}=0$ in the theorem 3, we get:

If

$$h(p,\ b)=H_{n_1,n_2}\ [f(x,\ y):p,\ q]$$

and

$$F(p,q)\doteqdot\left[(xy)^N h(x^{\frac{r_1}{s_1}},\ y^{\frac{r_2}{s_2}})\right],$$

then

$$(5.3)\quad F(p,\ q)=\frac{(2\pi)^{1-r_1-r_2}}{p^{N+\frac{r_1}{2s_1}}q^{N+\frac{r_2}{2s_2}}}\int_0^\infty\int_0^\infty\sqrt{st}\,\phi(p,\ q,\ s,\ t)f(s,\ t)dsdt,$$

where

$$\phi(p,\ q,\ s,\ t)=(2r_1)^{N+\frac{r_1}{2s_1}+\frac{1}{2}}(2r_2)^{N+\frac{r_2}{2s_2}+\frac{1}{2}}$$
$$\times G^{s_1,2r_1}_{2r_1,2s_1}\left(\left(\frac{2r_1}{p}\right)^{2r_1}\left(\frac{s}{2s_1}\right)^{2s_1}\middle|\begin{matrix}\Delta\left(2r_1,\ -N-\frac{r_1}{2s_1}\right)\\ \Delta\left(s_1,\pm\frac{n_1}{2}\right)\end{matrix}\right)$$
$$\times G^{s_2,2r_2}_{2r_2,2s_2}\left(\left(\frac{2r_2}{q}\right)^{2r_2}\left(\frac{t}{2s_2}\right)^{2s_2}\middle|\begin{matrix}\Delta\left(2r_2,\ -N-\frac{r_2}{2s_2}\right)\\ \Delta\left(s_2,\pm\frac{n_2}{2}\right)\end{matrix}\right),$$

the Hankel transform of $|f(x,\ y)|$ and Laplace transform of $|(xy)^N h(x^{\frac{r_1}{s_1}},y^{\frac{r_2}{s_2}})|$ exist, the integral in (5.3) is absolutely convergent, r_1, r_2, s_1 and s_2 are positive integers and

$$R\left(N+\frac{r_1}{2s_1}+\frac{r_1n_1}{s_1}+1\right)>0,\quad R(p,\ q)>0,$$

$$R\left(N+\frac{r_2}{2s_2}+\frac{r_2n_2}{s_2}+1\right)>0.$$

6. THEOREM 4.

If

$$h(p,\ q)=H_{n_1,n_2}[f(x,\ y):p,\ q],$$

and

$$F(p,q)=G\left[(xy)^N h(x^{-\frac{r_1}{s_1}},\ y^{-\frac{r_2}{s_2}})\right],$$

then

$$\text{(6.1)}\quad F(p,\ q)=\frac{(2\pi)^{1-r_1-r_2}}{p^{N-\frac{r_1}{2s_1}}q^{N-\frac{r_2}{2s_2}}}\int_0^\infty\int_0^\infty\sqrt{st}\,\phi(p,\ q,\ s,\ t)f(s,\ t)dsdt,$$

where

$$\phi(p,\ q,\ s,\ t)=(2r_1)^{a_{m+1}-\sum_{i=1}^m b_i+N-\frac{r_1}{2s_1}+\frac{1}{2}}(2r_2)^{c_{n+1}-\sum_{i=1}^n d_i+N-\frac{r_2}{2s_2}+\frac{1}{2}}$$
$$\times G^{2r_1m+2r_1+s_1,0}_{2r_1m,2r_1m+2r_1+2s_1}\left(\left(\frac{p}{2r_1}\right)^{2r_1}\left(\frac{s}{2s_1}\right)^{2s_1}\middle|\begin{matrix}\Delta\left(\left(2r_1,\ a_m+b_m+N-\frac{r_1}{2s_1}+1\right.\right.\\ \Delta\left(\left(2r_1,a_{m+1}+N-\frac{r_1}{2s_1}+1\right)\right),\Delta\left(s_1,\pm\frac{n_1}{2}\right)\end{matrix}\right)$$
$$\times G^{2r_2n+2r_2+s_2,\ 0}_{2r_2n,\ 2r_2n+2r_2+2s_2}\left(\left(\frac{q}{2r_2}\right)^{2r_2}\left(\frac{t}{2s_2}\right)^{2s_2}\middle|\begin{matrix}\Delta\left(\left(2r_1,c_n+d_n+N-\frac{r_2}{2s_2}+1\right)\right)\\ \Delta\left(\left(2r_2,c_{n+1}+N-\frac{r_2}{2s_2}+1\right)\right),\Delta\left(s_2,\pm\frac{1}{2}n_2\right)\end{matrix}\right),$$

the Hankel transform of $|f(x, y)$ *and Meijer-Laplace transform of* $|(xy)^N h(x^{-\frac{r_1}{s_1}}, y^{-\frac{r_2}{s_2}})|$ *exist, the integral in* (6.1) *is absolutely convergent;* r_1, r_2, s_1 *and* s_2 *are positive integers and*

$$R\left(N-\frac{r_1}{s_1}+a_j+1\right)>0,\quad j=1,\ \cdots\cdots,\ m+1,\quad R(p)>0,\quad |\arg\ p|<\frac{\pi}{2},$$

$$R\left(N-\frac{r_2}{s_2}+c_j+1\right)>0,\quad i=1,\ \cdots\cdots,\ n+1,\quad R(q)>0,\quad |\arg\ q|<\frac{\pi}{2}.$$

The proof of this theorem is similar to that of theorem 3. After changing the order of integration, the identity [1, p. 209, (9)] is used and then the inner double integral is evaluated with the help of (2.1).

(a) Particular case:

Substituting $b_i=0$, $i=1, \cdots\cdots, m$, $a_{m+1}=0 : d_j=0$, $j=1, \cdots\cdots, n$, $c_{n+1}=0$ in the theorem 5, we obtain:

If

$$h(p, q)=H_{n_1, n_2}[f(x,\ y)p,\ q],$$

and

$$F(p,\ q) \doteqdot (xy)^N h(x^{-\frac{r_1}{s_1}},\ y^{-\frac{r_2}{s_2}}),$$

then

$$(6.2)\quad F(p,\ q)=\frac{(2\pi)^{1-r_1-r_2}}{p^{N-\frac{r_1}{2s_1}}\, q^{N-\frac{r_2}{2s_2}}}\int_0^\infty\int_0^\infty \sqrt{st}\,\phi(p,\ q,\ s,\ t)f(s,\ t)dsdt,$$

where

$$\phi(p,\ q,\ s,\ t)=(2r_1)^{N-\frac{r_1}{2s_1}+\frac{1}{2}}(2r_2)^{N-\frac{r_2}{2s_2}+\frac{1}{2}}$$

$$\times G^{2r_1+s_1,0}_{0,2r_1+2s_1}\left(\left(\frac{p}{2r_1}\right)^{2r_1}\left(\frac{s}{2s_1}\right)^{2s_1}\middle|\begin{matrix}-\\ \Delta\left(2r_1, N-\frac{r_1}{2s_1}+1\right),\ \Delta\left(s_1, \pm\frac{1}{2}n_1\right)\end{matrix}\right)$$

$$\times G^{2r_2+s_2,0}_{0,2r_2+2s_2}\left(\left(\frac{q}{2r_2}\right)^{2r_2}\left(\frac{t}{2s_2}\right)^{2s_2}\middle|\begin{matrix}-\\ \Delta\left(2r_2, N-\frac{r_2}{2s_2}+1\right),\ \Delta\left(s_2, \pm\frac{1}{2}n_2\right)\end{matrix}\right),$$

the Hankel transform of $|f(x, y)|$ and the Laplace transform of $|(xy)^N h(x^{-\frac{r_1}{s_1}}, y^{-\frac{r_2}{s_2}})|$ exist, the integral in (6.2) is absolutely convergent; r_1, r_2, s_1, s_2 are positive integers

$$R\left(N-\frac{r_i}{s_i}+1\right)>0,\ i=1,\ 2,\cdots\cdots\cdots,\ R(p,\ q)>0.$$

Similarly, choosing the parameters suitably as in (B) and (C) of section 1, we get the corresponding relations for Hankel and Meijer, Hankel and Varma transforms of two variables.

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