INVARIANT SUBMANIFOLDS OF CODIMENSION 2 IN A LOCALLY PRODUCT RIEMANNIAN MANIFOLD

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It is well known that submanifolds of codimension 2 in an almost complex manifold are not in general almost complex. On the other hand invariant submanifolds of codimension 2 in an almost complex manifold are also almost complex and invariant submanifolds of codimension 2 in a (normal) contact Riemannian manifold are so also [6], [7].

In this paper, we shall prove that invariant submanifolds of codimension 2 in a locally product Riemannian manifold are also locally product Riemannian manifold.

In § 1 we give definition of a locally product Riemannian manifold by the almost product structure tensors point of view. In § 2 we give induced structures on submanifolds of codimension 2 in our manifold by devices similar to [1]. In § 3 we prove that the invariant submanifolds of codimension 2 in our manifold is also locally product. In § 4 we show non-existence of invariant totally umbilical submanifold of codimension 2, of non-zero mean curvature.

1. Locally product Riemannian manifolds.

We shall now recall definition of locally product Riemannian manifold for the later use. On an $(n+2)$-dimensional Riemannian manifold M, if there exists a tensor field F of type (1, 1) such that

(1.1) $$F^2=I,$$

(1.2) $$G(FX, FY)=G(X, Y),$$

(1.3) $$\nabla_X F=0,$$

where I denotes the identity tensor of type (1, 1) and ∇ the Riemannian connection determined by G, then the manifold M is called a locally product Riemannian manifold and the tensor field F defined by (1.1) is called an almost product structure.

We see that the matrix (F) has eigenvalues $+1$ and -1, and assume that $+1$ appears p times and -1 appears q times (so that $p+q=n+2$) among the eigenvalues of (F), then we have

(1.4) $$\text{trace } F=p-q.$$

In this case, the locally product Riemannian manifold M is locally the product $M^p \times M^q$ of two manifolds.

A tensor field Φ of type $(0,2)$ defined by

$$\Phi(X,Y)=G(FX,Y), \tag{1.5}$$

for any two vector fields X and Y is symmetric. i.e.,

$$\Phi(X,Y)=\Phi(Y,X). \tag{1.6}$$

2. Submanifolds of codimension 2 in a locally product Riemannian manifold.

Let N be a submanifold of codimension 2 imbedded in an $(n+2)$—dimensional locally product Riemannian manifold M with almost product structures (F,G). Thus, if i denotes the imbedding $N\to M$ and B the differential of i, then induced metric g on N is defined in term of the metric G on M by

$$g(X,Y)=G(BX,BY) \tag{2.1}$$

for any tangent vector fields $X$and Y on N.

We assume that the normal bundle of N is orientable, that is, there exists two unit vector fields C and D normal to $i(N)$ and mutually orthogonal, then we have

$$G(C,C)=1,\quad G(C,D)=0,\quad G(D,D)=1,\quad G(BX,C)=0,\quad G(BX,D)=0. \tag{2.2}$$

It is easy to see ([1], [7]) that we can define a tensor field f of type $(1,1)$, the vector fields E and A, 1-forms λ and μ, and scalar fields r,s and t on N by

$$FBX=BfX+\lambda(X)C+\mu(X)D, \tag{2.3}$$

$$FC=BE+rC+sD, \tag{2.4}$$

$$FD=BA+sC+tD. \tag{2.5}$$

PROPOSITION 1. *f,E,A,λ,μ,r,s,t satisfy*

$$f^2=I-\lambda\otimes E-\mu\otimes A,\qquad \lambda\cdot f=-r\lambda-s\mu,\qquad \mu\cdot f=-s\lambda-t\mu, \tag{2.6}$$

$$fE=-rE-sA,\qquad \lambda(E)=1-r^2-s^2,\qquad \mu(E)=-s(r+t), \tag{2.7}$$

$$fA=-sE-tA,\qquad \lambda(A)=-s(r+t),\qquad \mu(A)=1-s^2-t^2. \tag{2.8}$$

PROOF. Transforming again the both members of (2.3) by F, we have

$$BX=F^2BX=B(f^2+\lambda(X)E+\mu(X)A)+(f\lambda(X)+r\lambda(X)+s\mu(X))C$$
$$+(f\mu(X)+s\lambda(X)+t\mu(X))D.$$

Comparing tangential and normal parts we obtain the results (2.6). Similarly computing F^2C and F^2D, we have the relations (2.7) and (2.8) respectively.

PROPOSITION 2. *If $rt-s^2\neq 0$, then f is non-singular.*

PROOF. Suppose that $fX=0$, then $FBX=\lambda(X)C+\mu(X)D$, and hence

$$BX=F^2BX=B(\lambda(X)E+\mu(X)A)+(r\lambda(X)+s\mu(X))C+(s\lambda(X)+t\mu(X))D,$$

which yields $\lambda(X)=0$ and $\mu(X)=0$ for $rt-s^2\neq 0$. Since X is tangent to N and $BX=0$, then we have $X=0$.

PROPOSITION 3. *The tensor f defines an almost product structure on N if and only if $r+t=0$, $r^2+s^2=1$.*

PROOF. If $r+t=0$ and $r^2+s^2=1$, then from (2.7) and (2.8) we have $\lambda(E)=\mu(E)=0$ and $\lambda(A)=\mu(A)=0$, from which we get $E=A=0$. Hence by (2.6), f is an almost product structure on N.

Conversely, if f is an almost product structure, then from (2.6) we have $E=A=0$, and from (2.7) and (2.8) we obtain

$$r^2+s^2=1, \qquad s(r+t)=0, \qquad s^2+t^2=1.$$

If $s=0$, then from (2.6) we get $\lambda\cdot f=\pm\lambda$ and $\mu\cdot f=\pm\mu$, from which $f=\pm I$. This contradicts the fact that f is a non-trivial almost product structure. Thus we have $r+t=0$ and $r^2+s^2=1$.

PROPOSITION 4. *The induced metric g on N satisfy*

$$g(X,Y)=g(fX,fY)+\lambda(X)\lambda(Y)+\mu(X)\mu(Y), \tag{2.9}$$

$$g(fX,Y)=g(X,fY),$$

$$g(X,E)=\lambda(X), \qquad g(X,A)=\mu(X), \tag{2.10}$$

$$g(E,E)=1-r^2-s^2, \quad g(E,A)=-s(r+t), \quad g(A,A)=1-s^2-t^2. \tag{2.11}$$

PROOF. $$g(X,Y)=G(BX,BY)=G(FBX,FBY)$$
$$=g(fX,fY)+\lambda(X)\lambda(Y)+\mu(X)\mu(Y),$$
$$g(X,E)=G(BX,BE)=G(BX,FC-rC-sD)=G(FBX,C)=\lambda(X),$$
$$g(E,E)=G(BE,BE)=G(FC-rC-sD,\ FC-rC-sD)=1-r^2-s^2.$$

Similarly, we have the remaining results.

From (2.11) we immediately obtain that the induced vectors E and A are non-zero if and only if $r^2+s^2\neq 1$ and $s^2+t^2\neq 1$ respectively.

If we denote by the $\tilde{\nabla}$ the covariant differentiation with respect to G, then we have the equations of Gause-Weingarten

$$(\tilde{\nabla}_{BX}B)Y=h(X,Y)C+k(X,Y)D, \tag{2.12}$$

$$\tilde{\nabla}_{BX}C=-BHX+l(X)D, \qquad \tilde{\nabla}_{BX}D=-BKX-l(X)C,$$

where h and k are the second fundamental forms, and H and K are the corresponding Weingarten maps, and l is the third fundamental form.

Since the enveloping manifold M is a locally product, taking account of (1.3) we have

$$\tilde{\nabla}_{BX}FBY=h(X,Y)BE+k(X,Y)BA+(h(X,Y)r+k(X,Y)s)C$$
$$+(h(X,Y)s-k(X,Y)r)D+FB(\nabla_X Y).$$

On the othere hand

$$\tilde{\nabla}_{BX}FBY=\tilde{\nabla}_{BX}(BfY+\lambda(Y)C+\mu(Y)D)$$
$$=B((\nabla_X f)Y-\lambda(Y)HX-\mu(Y)KX+((\nabla_X\lambda)Y+h(X,Y)-l(X)\mu(Y))C$$
$$+((\nabla_X\mu)Y+k(X,Y)+l(X)\lambda(Y))D+\lambda(\nabla_X Y)C+\mu(\nabla_X Y)D,$$

where $\nabla_X Y$ denotes the component of $\tilde{\nabla}_{BX}BY$ tangent to N. Therefore, using (2.3) and comparing tangential and normal parts we have

(2.13) $$(\nabla_X f)Y=h(X,Y)E+k(X,Y)A+\lambda(Y)HX+\mu(Y)KX,$$

(2.14) $$h(X,fY)=rh(X,Y)+sk(X,Y)-(\nabla_X\lambda)(Y)+l(X)\mu(Y),$$

(2.15) $$k(X,fY)=sh(X,Y)+tk(X,Y)-(\nabla_X\mu)(Y)-l(X)\lambda(Y).$$

The equation (2.13) gives us an expression for the covariant derivative of f, clearly N is totally geodesic then f is covariant constant. More genearlly we prove

PROPOSITION 5. *Let N be a submanifold of codimension 2 in M, if $r+t=0$, $r^2+s^2=1$, then f is covariant constant.*

PROOF. If $r+t=0$ and $r^2+s^2=1$, by virtue of (2.11) we have $E=A=0$, and from (2.10) we have $\lambda(X)=\mu(X)=0$. Thus we get $\nabla_X f=0$.

3. Invariant submanifolds in a locally product Riemannian manifold.

We now assume that the tangent space of the submanifold N of codimension 2 in a locally product Riemannian manifold M is invariant under the action of the almost product structure tensor F of M, and such a submanifold an invariant submanifold.

For an invariant snbmanifold N, we have

(3.1) $$FBX=BfX,$$

that is

(3.2) $$\lambda=0, \qquad \mu=0.$$

in (2. 3). Hence from (2. 7) and (2. 8) we get

(3. 3) $$r^2+s^2=1, \qquad s(r+t)=0, \qquad s^2+t^2=1,$$

and from (2. 11) we have $E=A=0$.

We see easily that there occur only following two cases. i. e., case I and case II for an invariant submanifold N in a locally product Riemannian manifold M.

Case I ; $s=0$ and $r^2=t^2=1(rt>0)$.

Substituting above into (2. 6), we have

$$\lambda\cdot f=\pm\lambda, \qquad \mu\cdot f=\pm\mu,$$

which imply that

(3. 4) $$f=\pm I.$$

In this case, the equation (2. 4) and (2. 5) can be written in the following

(3. 5) $$FC=\pm C, \qquad FD=\pm D. \qquad \text{(resp.)}$$

From (3. 4) and (3. 5) we get

(3. 6) $$F=\pm I.$$

This contradicts the fact that F is a non-trivial almost product structure over on M.

Case II ; $r+t=0$ and $r^2+s^2=1$.

In this case, from (2. 11) we have $E=A=0$. and from (2. 10) we get $\lambda=\mu=0$. Threfore the submanifold N is an invariant.

Thus we have

THEOREM 6. *In order that a submanifold N of codimension 2 in a locally product Riemannian manifold M be an invariant, it is necessary and sufficient that $t=-r$, $r^2+s^2=1$ in* (2. 4) *and* (2. 5).

For an invariant submanifold N, by the Theorem 6, the equations (2. 4) and (2. 5) can be written in the following

(3. 7) $$FC=rC+sD, \qquad FD=sC-rD. \qquad (r^2+s^2=1)$$

In this case, the transforms of C and D by F on the normal space at every point of N is a reflexion with respect to any line through the point.

Next, since N is an invariant submanifold we have from (2. 6)

(3. 8) $$f^2=I,$$

and from (2. 9)

(3. 9) $$g(X,Y)=g(fX,fY),$$

from (2. 13)

(3. 10) $$\nabla_X f=0.$$

Thus we see that an invariant submanifold of codimension 2 in a locally product Riemannian manifold is also a locally product.

On the other hand, taking account of (3.8), the matrix (f) has ± 1 as eigenvalues, and we assume that (f) has eigenvelue $+1$ of multiplicity p' and eigenvalue -1 of multiplicity q', then we have

$$\text{trace } f = p' - q'. \tag{3.11}$$

Let $X_1, X_2, \cdots, X_n$ be a orthonormal local basis on N. Then $n+2$ vector fields BX_1, BX_2, $\cdots$, BX_n, C, D are also orthonormal basis at every point of M, and from (3.1) and (3.7) we have

$$\begin{aligned} \text{trace } F &= G(FBX_i, BX_i) + G(FC, C) + G(FD, D) \\ &= g(fX_i, X_i) + r - r \\ &= \text{trace } f, \end{aligned}$$

from (1.4) and (3.11) we obtain

$$p - q = p' - q'.$$

Since the invariant submanifold N is of codimension 2 in a $(n+2)$-dimensional manifold M, that is, $p' + q' = n$, hence we have

$$p' = p - 1, \qquad q' = q - 1. \tag{3.12}$$

Thus we have

THEOREM 7. *The invariant submanifold N in a locally product Riemannian manifold $M = M^p \times M^q$ is a locally product Riemannian manifold $N = N^{p-1} \times N^{q-1}$ with induced structures (f, g).*

4. Invariant totally umbilical submanifold in a locally product Riemannian manifold.

We assume that the enveloping manifold M is a locally product Riemannian manifold and the invariant submanifold N of codimension 2 imbedded in M is *a* totally umbilical. In this case, the second fundamental forms of N has the form

$$h(X, Y) = \bar{h} g(X, Y), \qquad k(X, Y) = \bar{k} g(X, Y), \tag{4.1}$$

where $\bar{h} = (1/n)$ trace h and $\bar{k} = (1/n)$ trace k.

For an invariant submanifold N, the equations (2.14) and (2.15) become respectively

$$h(X, fY) = rh(X, Y) + sk(X, Y), \tag{4.2}$$

$$k(X, fY) = sh(X, Y) - rk(X, Y), \tag{4.3}$$

and from which

$$h(fX, fY) = h(X, Y), \qquad k(fX, fY) = k(X, Y). \tag{4.4}$$

Substituting (4.1) into the equations (4.2) and (4.3) respectively we have

$$\bar{h}g(X,fY)=(r\bar{h}+s\bar{k})g(X,Y), \tag{4.5}$$
$$\bar{k}g(X,fY)=(s\bar{h}-r\bar{k})g(X,Y),$$

from which we have

$$(\text{trace } f)\ \bar{h}=n(r\bar{h}+s\bar{k}), \tag{4.6}$$
$$(\text{trace} f)\bar{k}=n(s\bar{h}-r\bar{k}),$$

and taking use of $r^2+s^2=1$, we have

$$(\text{trace } f)^2(\bar{h}^2+\bar{k}^2)=n^2(\bar{h}^2+\bar{k}^2). \tag{4.7}$$

According to Theorem 7, trace $f \neq \pm n$, then (4.7) imply that $\bar{h}=\bar{k}=0$.

Thus we have

THEOREM 8. *An invariant totally umbilical submanifold of codimension 2 in a locally product Riemannian manifold is a totally geodesic.*

If $\bar{h}^2+\bar{k}^2\neq 0$, that the invariant submanifold N has non-zero mean curvature, then we have trace $f=\pm n$. This contradicts to Theorem 7.

THEOREM 9. *Let M be a locally product Riemannian manifold, there is no totally umbilical invariant submanifold of codimension 2 in M, of non-zero mean curvature*

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