**Inf-Preserving Functors from A to Ens**

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1. **Introduction.** Let \( I \) and \( A \) be an index category and a small category respectively. For each object \( A \) of \( A \) the constant diagram \( A: I \to \text{Ens} \) is defined by \( A_i(i) = A, \ A_i(k) = 1_A \) for each object \( i \) and map \( k: i \to j \) of \( I \). A lower bound \((A, u)\) of a diagram \( F: I \to \text{Ens} \) consists of an object \( A \) of \( A \) and a natural transformation \( u: A_i \to F_i \). The lower bound \((A, u)\) of \( F \) will be called the infimum of \( F \) if for every lower bound \((A', u')\) of \( F \) there exists a unique map \( a: A' \to A \) such that \( u(i)a = u'(i) \) for all objects \( i \) of \( I \), and we write it by \( \inf F = (A, u) \). A functor \( F: \text{Ens} \to \text{Ens} \) is called the inf-preserving functor or we say that it preserves the infimums if for every diagram \( F: I \to \text{Ens} \), \( \inf F = (A, u) \Rightarrow \inf (F, F') = (F(A), F \cdot u) \). An upper bound, the supremum of a diagram and the sup-preserving functors are also defined dually. We write the opposite category of \( A \) and the category of sets by \( A^\circ \) and \( \text{Ens} \) respectively. J. Lambeck [1] proved that \( A \) is embedded as a sup-dense subcategory into a sup-complete category \( A' \) of all functors from \( A^\circ \) to \( \text{Ens} \) and the embedding functor of \( A \) into the category \( A'' \) of all inf-preserving functors from \( A^\circ \) to \( \text{Ens} \) is sup-dense and sup-preserving. Further he proved that the category \( A'' \) is inf-complete. The purpose of this note is to prove that the opposite \( A''^\circ \) of the category \( A'' \) of all inf-preserving functors from \( A \) to \( \text{Ens} \) is sup-complete and it is inf-complete if for any diagram \( \Theta \) with \( \inf \Theta = (z, t) \), \( t \) is a natural equivalence.

Throughout this note we assume that every diagram has the small index category.

2. **Inf-preserving functors.** Let \( \{a\} \) be a typical one element set and \( T: A \to \text{Ens} \). We may associate with the element \( x \) of \( T(A) \) for all \( A \) of \( A \) the map \( x: \{a\} \to T(A) \) such that \( x(a) = x \). The following lemmas will be stated whose proofs are to be found in [1] and [3] respectively.

**Lemma 1.** For any object \( A \) of \( A \) the functors \( [A, \ ]: A \to \text{Ens} \) and \( [\ , A]: \text{Ens} \to \text{Ens} \) preserves infimums.

**Lemma 2.** Let \( T \) and \( T' \) be two functors from \( A \) to \( B \) and \( \eta: T \to T' \) a natural equivalence. Then \( T \) preserves infimums if and only if \( T' \) does.

**Proposition 1.** The functor \( T: A \to B \) preserves infimums if and only if \( [B, T(\ )]: A \to \text{Ens} \) preserves infimums for all \( B \) in \( B \).

**Proof.** Assume that \( T \) preserves infimums. The functor \( [B, T(\ )] \) arises by composition from the inf-preserving functor \( T: A \to B \) and the functor \( [B, \ ]: B \to \text{Ens} \). But the functor \( [B, \ ] \) preserves infimum by the lemma 1. Hence it also preserves infimums. Conversely, assume that the functor \( [B, T(\ )] \) preserves infs for all \( B \) in \( B \). Let \( D: I \to A \) be a diagram in \( A \) with \( \inf D = (A, u) \). Then the infimum of the diagram \( [B, T(\ )] \circ D: I \to \text{Ens} \) is \((B, T(A)), v)\), where \( v(i) = [B, T(u(i))] \) for each \( i \) of \( I \). Let \( t(i): B \to T(D(i)) \) be natural in \( i \) of \( I \). We associate with \( t(i) \) mapping \( i(i): \{a\} \to [B, T(D(i))] \). Hence there exists a unique map \( g: \{a\} \to [B, T(A)] \) such that \( t(i) = v(i) \cdot g \). Since \( i(i)(a) = v \)

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we have a unique element \( g(\phi) \) of \([B, T(A)]\) such that \( t(\phi) = v(\phi)g(\phi) = T(u(\phi)) \cdot g(\phi) \). Hence \( \inf TD = [T(A), T \cdot u] \).

**Proposition 2.** Let \( T : A \to C \) be an embedding functor and \( C_0 \) be the subcategory of \( C \), consisting of all objects \( C \) in \( C \) such that the functor \([C, T()] : A \to \text{Ens}\) preserves infimums. Then

(i) the image \( T(A) \) of \( T \) is contained in \( C_0 \).

(ii) \( C_0 \) is the largest subcategory of \( C \) such that the induced embedding functor \( A \to T(A) \to C_0 \) preserves infimums.

(iii) Every supremum of any diagram \( A : J \to C \) in \( C \) is contained in \( C_0 \).

(iv) For any diagram \( \Theta : K \to C_0 \) with \( \inf \Theta = (C', \tau) \) in \( C \) if \( \tau \) is a natural equivalence then \( C' \) is contained in \( C_0 \).

**Proof.** (i) Since \( T \) is an embedding functor, we have a natural equivalence \( \mu : [A, \_] \cong [T(A), T(\_)] \) for each object \( A \) of \( A \), where \([A, \_] \) and \([T(A), T(\_)] \) are two functors from \( A \) to \( \text{Ens} \). The functor \([A, \_] \) preserves infimums by the lemma 1. Hence the functor \([T(A), T(\_)] \) preserves infimums, by the lemma 2. Therefore \( T(A) \in C_0 \) and \( T(A) \subseteq C_0 \).

(ii) Let \( A \to T(A) \to C_0 \) be an inf-preserving induced embedding functor. By the proposition 1, for each object \( C' \) of \( C' \), the functor \([C', T(\_)] \) preserves infimums. Hence \( C' \) is contained in \( C_0 \).

(iii) Let \( d : J \to C_0 \) be any diagram with \( \sup d = (C, v) \) in \( C \). Assume that \( D : I \to A \) is a diagram with \( \inf D = (A, u) \) then \( \inf T \cdot D = ((T(A), T \cdot u)) \) in \( C_0 \). Let \( g : C \to T(D) \) be a natural transformation, then we have the map \( g(i)v(j) \) in \( C \) such that

\[
\begin{align*}
A(j) & \xrightarrow{v(j)} C \\
& \downarrow g(i) \\
& \rightarrow TD(i)
\end{align*}
\]

Hence there exists a unique map \( S(j) : A(j) \to T(A) \) such that

\[
\begin{align*}
A(j) & \xrightarrow{v(j)} C \\
S(j) & \downarrow f \\
T(A) & \rightarrow TD(i)
\end{align*}
\]

Since \( T(u(i)) \cdot S(j) : A(j) \to TD(i) \) is a natural in \( i \in I \), so does \( S(j) \) in \( j \in J \). Hence there exists a unique map \( f : C \to T(A) \) such that \( S(j) = f \cdot v(j) \). Therefore \( T(u(i)) \cdot S(j) = T(u(i))f \cdot v(j) = g(i)v(j), \) hence \( g(i) = T(u(i))f \), that is, \( T(A) \) is the infimum in \( C_0 \cup \{C\} \), so that \( C_0 \cup \{C\} = C_0 \) by (ii).

(iv) Consider any diagrams \( \Theta : K \to C_0 \) with \( \inf \Theta = (C', \tau) \) in \( C \), where \( \tau \) is a natural equivalence and \( D : I \to A \) with \( \inf D = (A, u) \). Then \( T(A) \to TD(i) \) is a natural in \( i \in I \). Let \( x(i) : C \to TD(i) \) be a natural in \( i \in I \), for each \( k \) of \( K \) then we have a natural map \( x(i)t^{-1}(k) : \Theta(k) \to TD(i) \) in \( i \) of \( I \). There exists a unique map \( y(k) : \Theta(k) \to T(A) \)
such that \( T(u(i)) \cdot y(k) = x(i) \cdot t^{-1}(k) \). Hence \( T(u(i)) \cdot y(k) t(k) = x(i) \).

\[
\begin{array}{c}
\Theta(k) \\
T(u(i)) \\
T(A) \\
\end{array} \quad \begin{array}{c}
t^{-1}(k) \\
y(k) \\
x(i) \\
\end{array} \quad \begin{array}{c}
C' \\
T(D(i)) \\
\end{array}
\]

commutes.

By (ii), \( C' \) must be in \( C_0 \).

3. Category of inf-preserving functors. Let \( A \) be any small category. We shall write \([A, \text{Ens}]_{\text{inf}}\) for the category of all inf-preserving functors from \( A \) to \( \text{Ens} \).

**Lemma 3.** For each object \( C \) of \( C \) and each functor \( S: C \to \text{Ens} \), there is a bijection \( \psi: S(C) \cong [h, S] \), where \( h \) is a functor \([C, \ ]:\ C \to \text{Ens}\) and the map \( \psi \) is defined for \( x \in S(C) \) and \( f \in h(A) \) for \( A \) of \( C \) as \( \psi(x)(f) = (S(f))(x) \), [2].

By the lemma 1 the canonical embedding \( H: A \to [A, \text{Ens}]^\circ \) induce the embedding of \( A \) into \([A, \text{Ens}]_{\text{inf}}\). Using the proposition 2, we shall show the following theorem.

**Theorem.** (1) The category \([A, \text{Ens}]^\circ_{\text{inf}}\) is sup-complete.

(2) For any diagram \( \Theta \) with \( \inf \Theta = (z, t) \) in \([A, \text{Ens}]^\circ\) if \( t \) is a natural equivalence then \([A, \text{Ens}]_{\text{inf}}\) is inf-complete.

**Proof.** (1) Let \( B \) be the category of all functors \( T: A \to \text{Ens} \) in \([A, \text{Ens}]^\circ\) such that \([T, H(\ )]\) preserves infimums. By the lemma 3, \( T \cong [T, H(\ )] \). Therefore we have \([A, \text{Ens}]_{\text{inf}} = B \) by the lemma 2. Hence the category \( B \) is sup-complete by (iii) of the proposition 2.

(2) Since \([A, \text{Ens}]_{\text{inf}} = B \), by (iv) o proposition 2 it follows that \([A, \text{Ens}]_{\text{inf}}\) is inf-complete.

**References**


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