A note on the well-formed formulas of a pure functional calculus of the first order

Joseph T. Ryan

Abstract

The purpose of this note is to use primitive recursive functions and predicates similar to those presented by Davis [2] to show the well-known result first proved by Godel in [3] that the well-formed formulas of a pure functional calculus form a recursive set. The technique shown in this paper immediately suggests a unique approach to Theorem proving using a computer.

1. Definitions

A pure functional calculus of the first order, \( F^\alpha \), is defined by Church [1] to be the logistic system having as its primitive symbols the eight improper symbols

\[
\text{\(\Rightarrow\)} \sim (\, \, ) \ V
\]

and the infinite list of individual variables

\[ x \ y \ z \ x_1 \ y_1 \ z_1 \ x_2 \ y_2 \ z_2 \ldots \]

Also some or all of the following, including either at least one of the infinite lists of functional variables or at least one functional constant, are included: the infinite list of propositional variables

\[ p \ q \ r \ s \ p_1 \ q_1 \ r_1 \ s_1 \ p_2 \ldots \]

and, for each positive integer \( n \), an infinite list of \( n \)-ary functional variables, namely, the infinite list of singularly functional variables

\[ F^1 \ G^1 \ H^1 \ F_1^1 \ G_1^1 \ H_1^1 \ F_2^1 \ldots \]

the infinite list of binary functional variables
including any number of individual constants, any number of singularly functional constants, binary functional constants, ternary functional constants, etc.

The formation rules of $F^\rightarrow$ are given by Church [1] as:

1. A propositional variable standing alone is a well-formed formula (henceforth referred to as $wff$).

2. If $f$ is an $n$-ary functional variable or an $n$-ary functional constant, and if $a_1, a_2, ..., a_n$ are individual variables or individual constants or both (not necessarily all different), then $f(a_1, a_2, ..., a_n)$ is a $wff$.

3. If $\Gamma$ is a $wff$ then $\sim\Gamma$ is a $wff$.

4. If $\Gamma$ and $\Delta$ are $wff$ then $[\Gamma \supset \Delta]$ is a $wff$.

5. If $\Gamma$ is a $wff$ and $a$ is an individual variable, then $(\forall a) \Gamma$ is a $wff$.

We also define a number of functions and predicates suggested by Davis (1958):

D1. The function $x \div y$ is defined by

$$x \div y = \begin{cases} x < y & \text{if } x \leq y \\ 0 & \text{if } x > y \end{cases}$$

D2. If $P(y, x^{(n)})$ is an $(n+1)$-ary predicate then

$$f(z, x^{(n)}) = \frac{\delta}{\gamma \rightarrow} P(y, x^{(n)})$$

is understood to be the $(n+1)$-ary total function that satisfies the equation

$$f(z, x^{(n)}) = \min_y \{ y \leq z \mid P(y, x^{(n)}) \}$$

where this is defined and

$$f(z, x^{(n)}) = 0$$

elsewhere.

D3. The function $Pr(n)$ is defined as the $n$th prime in order of magnitude, where we arbitrarily take the 0th prime equal to 0, namely

$$Pr(0) = 0$$
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\[ Pr(n+1) = \Phi_{y=0}^\{Prime(y) \land \exists y > Pr(n)\} \]

where Prime(x) states that x is a prime number.

D4. Let \( P(x_1, \ldots, x_n) \) be an n-ary predicate. Then, by the extension of \( P \), written

\[ \{x_1, x_2, \ldots, x_n \mid P(x_1, x_2, \ldots, x_n) \} \]

we shall mean set of all n-tuples \((a_1, a_2, \ldots, a_n)\) for which \( P(a_1, a_2, \ldots, a_n) \) is true.

D5. The predicate \( Gl \) is defined by the equation

\[ n \cdot Gl \cdot x = \Phi_{y=0}^\{(Pr(n)^y \mid x) \land \neg (Pr(n) \cdot y + 1) \mid x)\} \]

(If \( x \) is the Gödel number of \( M \), where \( M \) consists of the symbols \( T_1, T_2, \ldots, T_p \), then if \( 0 < n \leq p \), \( n \cdot Gl \cdot x \) is the number associated with \( T_n \), whereas if \( n = 0 \) or \( n > p \), then \( n \cdot Gl \cdot x = 0 \).)

D6. The function \( L(x) \) is defined by the equation

\[ L(x) = \Phi_{y=0}^\{(Gl \cdot y > 0) \land \bigwedge_{i=0}^{x} (y + i + 1) \cdot Gl \cdot x = 0\} \]

D7. The expression \((x/y)\) states that \( y \) divides \( x \) evenly.

2. The recursiveness of the wff of \( F^i \)

Using the above functions and predicates it is possible to demonstrate that the wff of \( F^i \) form a recursive set. We proceed to arithmetize the formal system as follows:

For the primitive symbols of \( F^i \), we make the correspondences:

\[
\begin{array}{cccccccc}
\vdash & \supset & \neg & ( & ) & \lor & V & \\
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9
\end{array}
\]

For the infinite list of individual variables (which can be denumerably infinite in number), we make the correspondences:
For the individual constants:
\[
\begin{array}{ccccccc}
  f & g & h & f_1 & g_1 & h_1 & f_2 \\
  13 & 13^2 & 13^3 & 13^4 & 13^5 & 13^6 & 13^7 \\
\end{array}
\]

For the propositional variables:
\[
\begin{array}{ccccccc}
  p & q & r & s & p_1 & q_1 & r_1 & s_1 \\
  17 & 17^2 & 17^3 & 17^4 & 17^5 & 17^6 & 17^7 & 17^8 \\
\end{array}
\]

For the singularly functional variables:
\[
\begin{array}{ccccccc}
  F^1 & G^1 & H^1 & F_1^1 & G_1^1 & H_1^1 & F_2^1 \\
  19 & 19^2 & 19^3 & 19^4 & 19^5 & 19^6 & 19^7 \\
\end{array}
\]

For the singularly functional constants:
\[
\begin{array}{ccccccc}
  P^1 & Q^1 & R^1 & S^1 & P_1^1 & Q_1^1 \\
  23 & 23^2 & 23^3 & 23^4 & 23^5 & 23^6 \\
\end{array}
\]

For the binary functional variables:
\[
\begin{array}{ccccccc}
  F^2 & G^2 & H^2 & F_1^2 & G_1^2 \\
  29 & 29^2 & 29^3 & 29^4 & 29^5 \\
\end{array}
\]

For the binary functional constants:
\[
\begin{array}{ccccccc}
  P^2 & Q^2 & R^2 & S^2 & P_1^2 & Q_1^2 \\
  31 & 31^2 & 31^3 & 31^4 & 31^5 & 31^6 \\
\end{array}
\]

For the \(n\)-ary functional variables:
\[
\begin{array}{ccccccc}
  F^n & G^n & H^n & F_1^n \\
  \Pr(7+n) & [\Pr(8+n)]^3 & [\Pr(8+n)]^3 & [\Pr(8+n)]^4 \\
\end{array}
\]

Finally, for the \(n\)-ary functional constants, we make the correspondences:
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\[
\begin{array}{cccc}
P^n & Q^n & R^n & S^n \\
Pr(8+n) & [Pr(8+n)]^2 & [Pr(8+n)]^3 & [Pr(8+n)]^4
\end{array}
\]

Thus, any formula in \( F^p \) will correspond a sequence of numbers, namely those corresponding to the successive symbols of the formula. For example, to \([(\forall x)F(x) \supset (\forall y)G(y)\)] will correspond the sequence

\[
2 \ 6 \ 9 \ 11 \ 8 \ 19 \ 6 \ 11 \ 8 \ 3 \ 5 \ 6 \ 9 \ 11^2 \ 8 \ 19^2 \ 6 \ 11^2 \ 8 \ 4
\]

We can make a unique number correspond to the formula by taking the product of the successive prime numbers (in their natural order) with powers equal to the numbers of the symbols (in the order in which they occur).

Thus to the formula above will correspond the number

\[
2^2 \ 3^6 \ 5^3 \ 7^11 \ 11^8 \ 13^{19} \ 17^6 \ 19^{11} \ 23^8 \ 29^3 \ 31^5 \ 37^6 \ 43^8 \ 47^{11^2} \ 53^8 \ 59^{18^2} \ 61^6 \ 67^{11^2} \ 71^8 \ 73^4
\]

We call this the Godel number of the formula, or, if \( M \) is an expression consisting of the symbols \( F_1, F_2, \ldots, F_n \), and we let \( a_1, \ldots, a_n \) be the integers associated with these symbols, then the Godel number of \( M \) is the integer

\[
r = \prod_{k=1}^{n} Pr(k)^{a_k}
\]

and we write \( gn(M) = r \)

If \( M \) is empty we write \( gn(M) = 1 \)

We will define \( Welf(x) \) to mean \( x \) is the Godel number of a well-formed formula.

We will now define five intermediate predicates which will correspond to the formation rules of \( F^p \), (1), (2), (3), (4), and (5), respectively.

\[\text{(1')} \ Prop(x) \text{ holds if and only if}
\]

\[
\min \{ \text{\text{[\( GL \) \( x \leq 17 \text{\leq}_{\text{max} \text{\( 17 \) \[ \text{\text{\( (L(x) = 1) \land (17m/l \text{\( GL \)) \( x \)\]})}\]}}\]})}\}
\]

\( x \) is the Godel number of a propositional variable standing alone.\]

\[\text{(2')} \ Func(x) \text{ holds if and only if}
\]
This states that $x$ is a Godel Number of any $n$-ary functional variable or constant standing alone (with its argument).

(3') C.V. Neg($x$) holds if and only if

$$[[\text{Welf}(y)] \land [x \leftarrow 2^x y]]$$
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(If \( r \) is wff then \( r \sim \) is wff)

\( (4') \) C.V. \( \text{Impl}(x) \) holds if and only if
\[
\begin{align*}
&[[\text{Welf}(y)] \land [\text{Welf}(z)] \land [x = 2^2 \ast y] \\
&[\text{Pr}[L(y) + 2]]^3 \ast z] \land [\text{Pr}[L(y) + L(z) + 3]]^4] \end{align*}
\]

\( (5') \) C.V. \( \text{Quant}(x) \) holds if and only if
\[
\text{Welf}(y) \land \left\{ \min_n \land [11y = n] \right\}
\]
\[
[ x = 2^2 \ast 3^2 \ast 5 \ast 7 \ast y \land [11m/n] ]
\]

We now make the definition:

\( \text{Welf}(x) \) holds if and only if
\[
[[\text{Prop}(x)] \lor [\text{Func}(x)] \lor [\text{C.V. Neg}(x)] \lor [\text{C.V. Impl}(x)] \lor [\text{C.V. Quant}(x)]]
\]

We note that \( \text{Welf}(x) \) is recursive. But this simply states that the wff of \( F^\beta \) form a recursive set.

References


The Total Systems Corporation,* Honolulu, Hawaii

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* Besides serving as President, The Total Systems Corporation, Mr. Ryan is also a Senior Staff Consultant to the Control Data Corporation in logic design and systems analysis.