EXPANSIONS OF GENERALISED HYPERGEOMETRIC FUNCTIONS

By R. S. Dahiya

1. Introduction. The generalized hypergeometric polynomial [4] has been defined by

\[ F_p(x) = x^{(\delta-1)n} \sum_{p+\delta}^{\infty} \left( \begin{array}{c} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{array} : \lambda \right) \left[ \begin{array}{c} \delta-n, a_1, \cdots, a_p \\ b_1, \cdots, b_q : \lambda \right] x^\lambda, \]

where the symbol \( \Delta(\delta, -n) \) represents the set of \( \delta \)-parameters: \( -\frac{n}{\delta}, -\frac{n+1}{\delta}, \ldots, -\frac{n+\delta-1}{\delta} \) and \( \delta, n \) are positive integers. The polynomial is in a generalized form which yields many known polynomials on specializing the parameters.

For ease in writing, we employ the contracted notation

\[ pF_q(x) = pF_q(a_p | x) = \sum_{r=0}^{\infty} \frac{(a_p)_r x^r}{(b_q)_r r!}. \]

Thus \( (a_p)_r \) is to be interpreted as \( \prod_{j=1}^p (a_j)_r \) and similarly for \( (b_q)_r \). \( (a_p, e_p) \) denotes \( (a_1, e_1), \ldots, (a_p, e_p) \).

In this paper we have established some expansions for Hypergeometric functions. Some particular cases have also been given with proper choice of parameter.

2. The Expansions. Results to be proved are

\[ \sum_{r=0}^{n} (-1)^n r \binom{n}{r} \frac{\Gamma(\mu+(\delta-1/2)n-m+r+1/2)}{\Gamma(\mu+(\delta-1/2)n-k-r-1)} \]

\[ \cdot \left( \begin{array}{c} \Delta(\delta, -n), a_p, \Delta(c, -\mu-(\delta-1/2)n+m+r) \\ b_q \end{array} : \lambda \right) \left[ \begin{array}{c} \Delta(\delta, -n), a_p, \Delta(c, -\mu-(\delta-1/2)n+k) \\ b_q, \Delta(c, -\mu-(\delta-1/2)n-m), \Delta(c, -\mu-(\delta-3/2)n+m) : \lambda \end{array} \right] \]

\[ \frac{\lambda(-1)^e}{c^e} \]
where \( R\left( \mu + (\delta - 1)n \pm m + \frac{1}{2} \right) > 0; \delta, n \) and \( c \) are positive integers.

\[
(2.2) \sum_{r=0}^{n} (-)^{n-r} c_r \frac{\Gamma\left( m - k - r + \frac{1}{2} \right)}{\Gamma(\mu + (\delta - 1)n - k - r + 1)} \cdot p + q + c F_{q+2c} \]

\[
\lambda(-1)^{c} \]

\[
\frac{\lambda(-1)^{c}}{c^c} \]

(2.3) \sum_{r=0}^{n} (-)^{n-r} c_r \frac{\Gamma\left( \mu + \left( \frac{\delta}{2} - 1 \right)n - m - r + \frac{1}{2} \right)}{\Gamma\left( \mu + \left( \frac{\delta}{2} - 1 \right)n - k + r + 1 \right)} \cdot \rho + \delta Second F_{q+2c} \]

\[
\lambda(-1)^{c} \]

\[
\frac{\lambda(-1)^{c}}{c^c} \]

where \( R\left( \mu + (\delta - 1)n \pm m + \frac{1}{2} \right) > 0; \delta, n \) and \( c \) are positive integers.

\[
(2.4) \sum_{r=0}^{n} (-)^{n-r} c_r \frac{\Gamma\left( m - k - r + \frac{1}{2} \right)}{\Gamma(\mu + (\delta - 1)n - k - r + 1)} \cdot p + q + c F_{q+2c} \]
Expansions of Generalised Hypergeometric Functions

\[ \left[ \Delta(\beta, -n), a_p, \Delta(c, \mu + (\beta - 1)n + m + \frac{1}{2}), \Delta(c, \mu + (\beta - 1)n - m + \frac{1}{2}) \right] \lambda \in \mathbb{C} \]

\[ b_q, \Delta(c, \mu + (\beta - 1)n - k - r + 1); \]

\[ \frac{\Gamma(m - k - n + \frac{1}{2}) \Gamma(\mu + \beta n - m + \frac{1}{2})}{\Gamma(\mu + (\beta - 1)n + \frac{1}{2}) \Gamma(\mu + (\beta - 1)n - m + \frac{1}{2})} \]

\[ \mu + (\beta - 1)n - k + 1 \]

\[ \mu + (\beta - 1)n + m + \frac{1}{2} \]

where \( R(\mu + (\beta - 1)n + m + \frac{1}{2}) > 0; \beta, n \) and \( c \) are positive integers.

3. Proofs: Since (Pathan, 1968; p.17)

\[ (3.1) \sum_{r=0}^{n} (-)^{n-r} C_r x^{-\frac{r}{2}} W_{k-\frac{1}{2}, \frac{1}{2}}(x) = \frac{\Gamma(m + n - k + \frac{1}{2})}{\Gamma(m - k + \frac{1}{2})}. \]

\[ \cdot W_{k-\frac{1}{2}, \frac{1}{2}}(x) = \frac{\Gamma(m + n - k + \frac{1}{2})}{\Gamma(m - k + \frac{1}{2})}. \]

we have

\[ (3.2) \int_{0}^{\infty} x^r f(x) \left( \sum_{r=0}^{n} (-)^{n-r} C_r x^{-\frac{r}{2}} W_{k+\frac{1}{2}, \frac{1}{2}}(x) \right) dx \]

\[ = \frac{\Gamma(m + n - k + \frac{1}{2})}{\Gamma(m - k + \frac{1}{2})} \int_{0}^{\infty} x^r f(x) \cdot W_{k-\frac{1}{2}, \frac{1}{2}}(x) dx \]

provided that the integrals involved exist. Now if we take

\[ (3.3) f(x) = x^{(\beta - 1)n - 1} e^{-\frac{1}{2} x} F_{pq} \left[ \Delta(\beta, -n), a_p; b_q; \lambda x^{-c} \right] \]

in (3.2) and evaluate the integrals involved there in with the help of the following results (Shah, 1969; p.484)

\[ (3.4) \int_{0}^{\infty} x^{\frac{1}{2} - 1} W_{k, m}(x) \left( x^{(\beta - 1)n} F_{pq} \left[ \Delta(\beta, -n), a_p; b_q; \lambda x^{-c} \right] \right) dx \]

\[ = A_{p+\beta+\epsilon} F_{q+2}\left[ b_q, \Delta(c, \mu + (\beta - 1)n + k); \Delta(c, \frac{1}{2} - \mu + (\beta - 1)n + m); \lambda(-1)^c \right], \]
where
\[ A = \frac{\Gamma\left(\mu + (\delta-1)n + m + \frac{1}{2}\right) \Gamma\left(\mu + (\delta-1)n - m + \frac{1}{2}\right)}{\Gamma(\mu + (\delta-1)n - k + 1)} \],

\[ R\left(\mu + (\delta-1)n \pm m + \frac{1}{2}\right) > 0 \] and \( \delta, n, c \) are positive integers. We get the recurrence relation (2.1) after some slight changes.

The remaining relations can be proved in a similar manner by using the following results:

\[
\sum_{r=0}^{\infty} \frac{\Gamma\left(m-k-r+\frac{1}{2}\right)}{\Gamma\left(m-k-n+\frac{1}{2}\right)} (-1)^{n-r} C_r W_{k+r, m}(x) = \frac{1}{x^{\frac{1}{2}}} W_{k+\frac{1}{2}, m-\frac{1}{2}}(x) \]

and

\[
\int x^{\nu-1} e^{-\frac{1}{2} x} W_{k,m}(x) \left( x^{(\delta-1)} \right) d\nu = A_{\nu} + \frac{1}{2} F_{q \nu} \]

where \( R\left(\mu + (\delta-1)n \pm m + \frac{1}{2}\right) > 0 \) and \( \delta, n, c \) are positive integers.

Use (3.5) and (3.4) to get (2.2).

Use (3.1) and (3.6) to get (2.3).

Use (3.5) and (3.6) to get (2.4).

4. particular cases: When \( \delta = c = \lambda = 1 \) in (2.3):

(a) with \( a_1 = n + 1 + \alpha + \beta + 1, \quad b_1 = 1 + \alpha, \quad b_2 = \frac{1}{2}, \) we obtain

\[
\sum_{r=0}^{n} (-1)^{n-r} C_r \frac{\Gamma\left(\mu + \frac{n}{2} - m - r + \frac{1}{2}\right)}{\Gamma\left(\mu + \frac{n}{2} - k - r + 1\right)} f_n^{(\alpha, \beta)}
\]

\[
\times \left( \begin{array}{c} a_2 \cdots a_r \mu + \frac{n}{2} - m + \frac{1}{2}, \mu + \frac{n}{2} - m - r - \frac{1}{2} \\ b_2 \cdots b_r \mu + \frac{n}{2} - k - r + 1; 1 \end{array} \right)
\]

\[
= \frac{\Gamma\left(m+n-k+\frac{1}{2}\right) \Gamma\left(\mu - m + \frac{n}{2} + \frac{1}{2}\right)}{\Gamma\left(m-k+\frac{1}{2}\right) \Gamma\left(\mu + \frac{n}{2} - k + 1\right)} f_n^{(\alpha, \beta)}
\]
\[
\begin{align*}
\times \left( a_2, \ldots, a_p, \mu + m + \frac{n}{2} + \frac{1}{2}, \mu - m - \frac{n}{2} - \frac{1}{2} \right) \\
\left( b_3, \ldots, b_q, \mu - k + \frac{n}{2} + 1 \right)
\end{align*}
\]

where \( R(\mu \pm m + \frac{n}{2} + \frac{1}{2}) > 0 \) and \( f_n^{(\alpha, \beta)} \) is a generalized Sister celine's polynomials [1].

(4.2) With \( p=0, q=1, b_1=1+\alpha, m=\alpha+\beta, \mu=\frac{n}{2} + \frac{1}{2} \), we obtain
\[
\sum_{r=0}^{n} (-)^{n-r} nC_r \frac{\Gamma(n-\alpha-\beta-r+1)}{\Gamma(n-k-r+3/2)} H_n^{\alpha,\beta} \left( n-\alpha-\beta-\gamma+1, \frac{n-k-r+3}{2}; 1 \right) = \frac{\Gamma(\alpha+\beta-k+1/2)}{\Gamma(\alpha+\beta-k+1/2)} \frac{\Gamma(n-k+3)}{\Gamma(n-k+1/2)} H_n^{\alpha,\beta} \left( 1-\alpha-\beta, \frac{n-k+3}{2}; 1 \right),
\]

where \( H_n^{\alpha,\beta} \) is a generalized Rice's polynomial [2].

(4.3) With \( p=0, q=1, m=-\mu-3n/2, b_1=\mu, k=-1/2 \); we get
\[
\sum_{r=0}^{n} (-)^{n-r} nC_r \frac{\Gamma(2\mu+2n-r+1/2)}{\Gamma(\mu+3n/2-r+3/2)} (-\mu-3n/2-1/2+r)_{2n} \cdot R_{2n}(\mu-3n/2-1/2+r, \mu; 1) = \frac{\Gamma(1-\mu-n/2)}{\Gamma(1-\mu-3n/2)} \frac{\Gamma(2\mu+n+1/2)}{\Gamma(\mu+n/2+3/2)} \mathcal{F}_2^{(\mu, \mu+n+1/2)} \left( \begin{array}{c} \mu-3n/2-1/2+r, \mu-n+1/2, 2\mu+n+1/2 \\ \mu, \mu+n+3/2 + 1/2 \end{array} ; 1 \right),
\]

where \( R_n(\beta, \tau; x) \) is a Bedient's polynomial [3, 297(1)]

(b) when \( c=2, \delta=1, \lambda=4, a_1=n+\alpha+\beta+1, b_1=1+\alpha, b_2=\frac{1}{2} \) in (2.1), we get

(4.4) \[
\sum_{r=0}^{n} (-1)^{n-r} \frac{\Gamma(\mu-m-r+n/2+1/2)}{\Gamma(\mu-k-r+n/2-1)} f_n^{(\alpha, \beta)}
\times \left[ a_2, \ldots, a_p, \mu - n/2 + k + r, 1 \right]
\times \left[ b_3, \ldots, b_q, (1/2 - \mu - n/2 - m, 1/2 - n/2 - \mu + m + 3) \right]
\]
\[
= \frac{\Gamma(m+n-k+1/2)}{\Gamma(m-k+1/2)} \frac{\Gamma(\mu-k+n/2+1/2)}{\Gamma(\mu-k+n/2+1)} f_n^{(\alpha, \beta)}
\]
\[
\times \left[ \begin{array}{c}
\left( a_2, \ldots, a_p, k-m-m/2, \\
\left( b_1, \ldots, b_q, 1/2-n-m-1/2, m+1/2-m+1/2 \right)
\end{array} \right]
\]

\[ R(m+n-k+1/2) > 0, R(m+n-k+1/2) \geq 0. \]

(4.5) When \( \delta = c = 2, p = 1, q = 2, a_1 = r - \beta, b_1 = r - \beta, b_2 = 1 - \beta + m, \lambda = 4 \) in (2.1),

We get

\[
\sum_{r=0}^{n} (-)^{n-r} \binom{n}{r} C_r \frac{\Gamma(\mu+3n/2-m-r+1/2)}{\Gamma(\mu+3n/2-k-r+1)} \frac{1}{5F6} \times \left[ \Delta(2, -n), r, 1-\beta-n, \Delta(2, 1/2-\mu-3n/2+m) : 1 \right]
\]

\[
= \frac{\Gamma(m+n-k+1/2)}{\Gamma(m-k+1/2)} \frac{\Gamma(\mu+n/2-m+1/2)}{\Gamma(\mu+3n/2-k+1)} \frac{1}{5F6} \times \left[ \Delta(2, -n), r, 1-\beta-n, \Delta(2, 1/2-\mu-3n/2+m) : 1 \right].
\]

where \( R(m+n-k) \geq 0, R(\mu+n/2-m+1/2) > 0 \) and \*\( \Delta(2, 1/2-\mu-3n/2+m) \)

\[
= \Delta(2, \frac{1}{2}-\mu-3n/2+m+r), \Delta(2, 1/2-\mu-3n/2+m+r).
\]

Iowa State University

REFERENCES


