The purpose of this note is to extend the trivial fact in the real line, \( \mathbb{R} \), that if \( f: \mathbb{R} \to \mathbb{R} \) and constants \( a, b > 0 \) such that
\[
f(y)\varepsilon = o(z^2) \quad \text{with} \quad a|y| < |z| < b|y|,
\]
then
\[
\lim_{y \to 0} \frac{|f(y)|}{|y|} = 0.
\]

The extension is a useful estimation in the calculus on Banach spaces as for example in the proof of the converse of Taylor's Theorem. \([1]\) \([2]\)

Let \( U \) be an open neighborhood of the origin in a Banach space \( E \) and let \( f: U \to \mathcal{L}_k(E, F) \) be a map from \( U \) into the space of bounded, symmetric \( k \)-linear maps of \( E \) into a Banach space \( F \). \( \mathcal{L}_k(E, F) \) is a closed subspace of the Banach space \( \mathcal{L}(E, F) \) of bounded, \( k \)-linear maps of \( E \) into \( F \), with the operator norm induced from that of \( E \) and \( F \), i.e., if \( A \in \mathcal{L}_k(E, F) \), then
\[
\|A\| = \sup\{|A(h_1, \ldots, h_k)|_F | h_i \in E, |h_i| \leq 1\}.
\]

We note that there is a norm-preserving isomorphism of \( \mathcal{L}_k(E, F) \) onto \( \mathcal{L}(E, \mathcal{L}(E, \ldots, \mathcal{L}(E, F)) \ldots) \), \( k \) times, under the identification which takes an element of the latter into the form given by
\[
A(y_1, \ldots, y_n) = ((\cdots(Ay_1)\cdots)\cdots y_{n-1})y_n.
\]

If \( A \) is in \( \mathcal{L}_k(E, F) \) we denote the value \( A(y_1, \ldots, y_n) \) by \( Ay_1\cdots y_n \). Also if \( y \) is in \( E \), the \( y^n \) means \((y, \ldots, y)\) \( n \) times. Hence
\[
A z^n_1 \cdots z^n_k = ((\cdots(Az^n_1)\cdots z^n_2)\cdots)z^n_k = A(z_1, \ldots, z_1, \ldots, z_k, \ldots, z_k).
\]

We adopt the following standard notation:
\[
A z^k = o(z^{k+\gamma}) \quad \text{where} \quad \gamma > 0, \quad \text{means} \quad \lim_{z \to 0} \frac{|A z^k|}{|z|^{k+\gamma}} = 0.
\]

**THEOREM.** Let \( \gamma \) be a non-negative real number. Let \( E \) and \( F \) be Banach spaces and \( U \) an open subset of \( E \). Let \( f: U \to \mathcal{L}_k(E, F) \). If there exist constants \( a, b > 0 \) such that \( f(y)z^k = o(z^{k+\gamma}) \) for \( a|y| < |z| < b|y| \) for \( z, y \in E \), then \( f(y) = o(y^\gamma) \).

**PROOF.** By induction on \( k \). For \( k = 1 \), we have \( f(y)z = o(z^{1+\gamma}) \). Given \( \varepsilon > 0, \delta > 0 \) such that
\[
\frac{|f(y)z|}{|z|^{1+\gamma}} < \frac{\varepsilon}{b^\delta} \quad \text{for} \quad |z| < b\delta.
\]
This implies
\[
\left| \frac{f(y) - z}{|z|} \right| \leq \frac{\varepsilon}{b} |z|^r < \varepsilon |y|^r \text{ since } |z| < b|y|, \text{ for } |y| < \delta.
\]

Taking supremum over \( z \leq \frac{b\delta}{2} \) we obtain

\[
\|f(y)\| < \varepsilon |y|^r \text{ for } |y| < \delta.
\]

Thus \( f(y) = o(y^r) \).

Assume that the theorem is true for \( k < n \). We first note that we can choose \( c, d > 0 \) \( a < c < d < b \), and \( a|y| < |z_1 + \lambda z_2| < b|y| \) for \( z_1 \) and \( z_2 \) with \( c|y| < |z_1| < d|y| \). \( i = 1, 2 \), provided \( \lambda \) is small enough, as it easily follows from

\[
|z_1| - |\lambda| |z_2| \leq |z_1 + \lambda z_2| \leq |z_1| + |\lambda| |z_2|.
\]

This enables us to substitute freely the element in \( o \) (element) estimates.

By multilinearity,

\[
f(y)(z_1 + \lambda z_2)^n = f(y)z_1^n + \lambda C^n_1 f(y)z_1^{n-1}z_2 + \cdots +
\]

\[
\lambda C^n_{n-1} f(y)z_1z_2^{n-1} + \lambda^n f(y)z_2^n = o(y^{n+r}).
\]

Since \( f(y)z_1^n = o(y^{n+r}) \) and \( f(y)z_2^n = o(y^{n+r}) \), we have

\[
\lambda C^n_1 f(y)z_1^{n-1}z_2 + \cdots + \lambda C^n_{n-1} z_1z_2^{n-1} = o(y^{n+1}).
\]

We now choose \( \lambda_i \neq \lambda_j \) for \( i \neq j \), \( i = 1, 2, \ldots, n-1 \), and each \( \lambda_i \) sufficiently small. Thus we have a system of equations in matrix form

\[
\begin{pmatrix}
\lambda_1 & \lambda_2 & \cdots & \lambda_1^{n-1} \\
\lambda_2 & \lambda_2 & \cdots & \lambda_2^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{n-1} & \lambda_{n-1} & \cdots & \lambda_{n-1}^{n-1}
\end{pmatrix}
\begin{pmatrix}
C_1^n f(y)z_1^{n-1}z_2 \\
C_2^n f(y)z_1^{n-2}z_2^2 \\
\vdots \\
C_{n-1} f(y)z_1z_2^{n-1}
\end{pmatrix}
= \begin{pmatrix}
o(y^{n+r}) \\
o(y^{n+r}) \\
\vdots \\
o(y^{n+r})
\end{pmatrix}
\]

Since \( \lambda_i \neq \lambda_j \), the above matrix is invertible. Hence \( f(y)z_1z_2^{n-1} = o(y^{n+r}) \). \( f(y)z_1^{n-1} = o(y^{n-1+r}) \).

By induction hypothesis, \( f(y)z_1 = o(y^r) \). Again by case \( k = 1 \), \( f(y) = o(y^r) \).

This concludes the proof.

\textbf{REFERENCES}
